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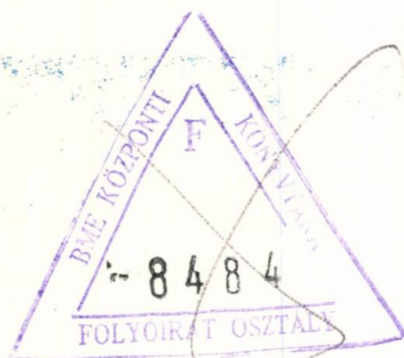
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ALGORITHMS OF ERROR DETECTION AND CORRECTION IN TREE-CODES

ADRIAN ATANASIU

Abstract

The main concern of this paper is one of the important problems we have already dealt with in the Coding Theory, that is decodification and error-correction, so continuing the results obtained in [1] și [3] regarding C-grammars and tree-codification.

The author presents decodification and error-correction in the tree-codes. So, the paper shows three algorithms of error detection and correction using particular properties of constructing C-grammars and tree-codifications. The methods are illustrated by numerous examples.

1. Introduction

When sending a message by a channel, it is inherent to appear disturbances which can lead to modifying the message. Considering that the number of disturbing signals does not go beyond a certain limit, we shall try to construct the code so that the message received could be correctly decoded.

Because there are two steps of decodification in fact, the errors could appear at two moments:

— for an easy transmission through the channel, every element of V_e is a vector of finite symbols, which according to Coding Theory could be marked $\{0, 1, \dots, q-1\}$; the alphabet $\{0, 1\}$ is used most frequently. It is possible that the sequence of the mentioned symbols to be intercepted would be so much disturbed that no element of V_e should appear. Then, at first decodification, this sequence is replaced by another one which contains an element of V_e following a certain rule; Coding Theory establishes that this word is — most probably — the closest in Hamming's metric to the word received.

After this step we could apply one of the error detection and correction algorithms we have already shown in this paper, in order to learn the correct sequence.

For a correct decodification of a character, we should take into account the following $l-1$ received characters which could give information on the error. Since the minimum constraint length of a tree codification is l , after l characters, any information on the error is lost.

For example, if we have the tree-codification given by the tree-structure in figure 1, it has $l = 3$.

We suppose, we found out that x_2 was possibly wrong. Such being the case the following two characters would give information on which of the possible paths through the tree must be followed.

We have already noticed that the third character is useless as it always accepts two solutions: for example it could accept both solution $y_2 x x_1 y_1$ — decoding 1001, and $x_2 y_3 x_3 y$ — decoding 0101.

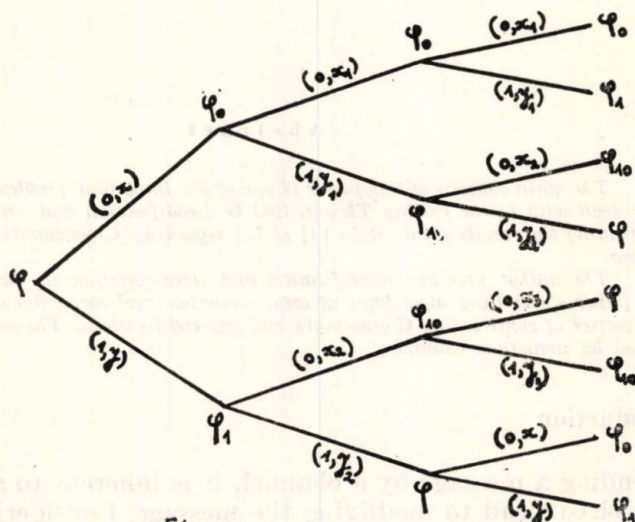


Figure 1.

Fig. 1

Besides, for a correct working of any algorithm it is necessary and sufficient to use the following property well defined by Coding Theory:

(1) In any sequence of t successive characters could appear at least $\left\lfloor \frac{t-1}{2} \right\rfloor$ errors.

This general property is graphically obvious: indeed, in order to establish the only path through the tree attached to the tree-code, it is enough to know half and one of the t successive nodes of the path.

2. Preliminary results

Definition: $G = (V_N, V_i, V_e, S, F)$ is C -grammar if the alphabet of terminals is composed by two finite nonempty sets V_i (input alphabet) and V_e (output alphabet), with the properties:

(a) for each pair $(A, a) \in V_N \times V_i$, there are two and only two productions of the form $A \rightarrow aBx \mid ax$, $x \in V_e$;

(b) for any $A, B \in V_N$ there is $\alpha \in V_i^+$, $w \in V_e^+$ with $A \Rightarrow^* \alpha B w$;

(c) two pairs of distinct productions from (a) have distinct x 's;

(d) F has no other productions.

Clearly, $L(G) = \{\alpha \tilde{w} \mid \alpha \in V_i^*, w \in V_e^*, S \Rightarrow^* \alpha \tilde{w}\}$

Hence, any word in $L(G)$ is a concatenation of two equal length sequences the input-word — uncodified, and the code-word send by the channel. From the construction will result for every $\alpha \in V_i^*$ there is $w \in V_e^*$, unique, with $\alpha\tilde{w} \in L(G)$.

For any $A \in V_N$ on define a mapping $\gamma_A: V_i^* \rightarrow V_e^*$ as follows: $\gamma_A(\alpha) = \gamma_A(a_1 a_2 \dots a_n) = w_1 w_2 \dots w_n$ (γ_A is a codification mapping) where $w_i \in V_e$ with $A \Rightarrow^* a_1 a_2 \dots a_n w_n \dots w_2 w_1$.

Definition: The one-to-one mapping $\varphi: A^* \rightarrow B^*$ is a sequential codification on (A, B) , A, B nonempty sets, $|A| \geq 2$, if

- (a) for any $\alpha, \beta \in A^*$, there is $w \in B^*$ with $\varphi(\alpha\beta) = \varphi(\alpha)w$;
- (b) $l(\varphi(\alpha)) = l(\alpha)$;
- (c) for $\alpha \in A^*$ there is $\alpha' \in A^*$ so that for any $\beta \in A^*$, $\varphi(\alpha\alpha'\beta) = \varphi(\alpha\alpha')\varphi(\beta)$.

For a sequential codification φ , we define the state of φ in the word α , that mapping $\varphi_\alpha: A^* \rightarrow B^*$ with the property: $\varphi(\alpha\beta) = \varphi(\alpha)\varphi_\alpha(\beta)$, ($\forall \beta \in A^*$). It's easy to see that φ_α is a sequential codification.

Let B^φ be the set of all states of a sequential codification φ ($\varphi \in B^\varphi$ because $\varphi = \varphi_\epsilon$). On B^φ we define the following equivalence relation R : $\varphi_\alpha R \varphi_\beta \Leftrightarrow \varphi_\alpha(u) = \varphi_\beta(u)$ ($\forall u \in A$). Let $A^\varphi = B^\varphi / R$. We choose the elements of A with the property

$$\varphi_\alpha \in A^\varphi \Leftrightarrow l(\alpha) = \min \{l(\beta) | \varphi_\beta R \varphi_\alpha\}. \text{ Obviously } \varphi \in A^\varphi.$$

Definition: A sequential codification φ is a tree-codification if

- (a) A^φ is a finite set
- (b) $\{\varphi_\alpha(A)\}_{\varphi_\alpha \in A^\varphi}$ is a partition of B .

Let G be a C-grammar. Because γ_A has the properties of a sequential codification, we say a sequential codification φ on (V_i, V_e) is achievable in C-grammar G with $A \in V_N$ if $\varphi R \gamma_A$.

We can prove that a sequential codification is a tree-codification if and only if it is achievable in a C-grammar.

By construction, the C-grammar associated with the tree-codification φ is $G = (A^\varphi, V_i, V_e, \varphi, F)$, where F contains only productions of the form $\varphi_\alpha \rightarrow a\varphi_\beta \varphi_\alpha(a) | a\varphi_\alpha(a)$, with $\varphi_\beta R_{\alpha a}$, $\varphi_\beta \in A^\varphi$.

Definition: A tree-code is a pair (φ, L) where φ is a tree-codification achievable in a C-grammar $G = (A^\varphi, V_i, V_e, \varphi, F)$ and

$$L = \{w | w \in V_e^*, \alpha\tilde{w} \in L(G), \alpha \in V_i^*\}$$

Algebraic convolutional codes are examples of tree-codes ([3]).

An important constant attached to the tree-code is l -minimal constraint length, defined as follows:

Let $\varphi_\alpha \Rightarrow_{k_\alpha} \beta_1 \varphi_\gamma w_1$, $\varphi_\alpha \Rightarrow_{k_\alpha} \beta_2 \varphi_\gamma w_2$ be two derivations with $\varphi_\alpha \varphi_\beta \in A^\varphi$, $\beta_1 \neq \beta_2$ and k_α minimal with this property.

Then $l = \min_{\varphi_\alpha \in A^\varphi} \{k_\alpha\}$

The case $l = 1$ can be assimilated to linear codes. In this paper we consider only the case $l > 1$.

3. Algorithms of error-detection and correction based on the predecessor and the successor functions.

The first algorithm refers to the very simple case when $l \geq 3$ and when (1) becomes:

(2) Any sequence of three successive characters of the code-word sent could accept one error at most.

We introduce separately this algorithm as it is totally different from the others and because its idea is very simple.

Definition: Let φ be a tree-codification achievable in a C-grammar $G = (A^\varphi, V_i, V_e, \varphi, F)$. We define the mappings S and $D: A^\varphi \rightarrow P(A^\varphi)$ as follow:

$$S(\varphi_\alpha) = \{\varphi_\beta \mid \varphi_\beta \rightarrow a \varphi_\alpha u \in F\}$$

$$D(\varphi_\alpha) = \{\varphi_\beta \mid \varphi_\alpha \rightarrow a \varphi_\beta u \in F\}$$

S is called predecessor function, while D -successor function.

Proposition: $l > 2$ if and only if $|D(\varphi_\alpha) \cap S(\varphi_\beta)| \leq 1$ for any φ_α and φ_β from A^φ .

Proof: „if”: We suppose $D(\varphi_\alpha) \cap S(\varphi_\beta) = \{\varphi_{\gamma_1} \varphi_{\gamma_2}\}$, $\gamma_1 \neq \gamma_2$. Then, in F there are the productions

$$\varphi_\alpha \rightarrow a_1 \varphi_{\gamma_1} u_1 \mid a_2 \varphi_{\gamma_2} u_2, \quad \varphi_{\gamma_1} \rightarrow a_3 \varphi_\beta u_3, \quad \varphi_{\gamma_2} \rightarrow a_4 \varphi_\beta u_4$$

Consequently, there will be two distinct words of length 2 which connect the states φ_α to φ_β ; hence it results — according to the definition — $l = 2$, contradiction.

„only if”: the case $l = 1$ has been eliminated from the initial hypotheses. We suppose $l = 2$; re-building viceversa the first part of the demonstration we obtain the result we wanted.

Example 1: In the tree-codification constructed above, we have

$$D(\varphi) = \{\varphi_0, \varphi_1\}, \quad S(\varphi) = \{\varphi_1, \varphi_{10}\}, \quad S(\varphi_{10}) = \{\varphi_1, \varphi_{10}\}$$

$$\text{hence } D(\varphi) \cap S(\varphi) = \{\varphi_1\} \quad D(\varphi) \cap S(\varphi_{10}) = \{\varphi_1\}$$

As we already know ([1], [2]), for a C-grammar G of a tree-codification φ , the codification of a sequence $\alpha \in V_i^*$ is $\varphi(\alpha) \in V_e^*$ obtained by the derivation $\varphi \Rightarrow^* \alpha \varphi(\alpha)$

In the absence of errors, decodification itself is very simple.

For each $u \in V_e$, in F there are two and only two productions which contain u :

$$\varphi_\alpha \rightarrow a \varphi_\beta u | au$$

Consequently we could define the mappings $\lambda, \mu: V_e \rightarrow A^\varphi$ as such:

$$\lambda(u) = \varphi_\alpha, \mu(u) = \varphi_\beta, \text{ and the mapping}$$

$$\psi: \lambda(V_e) \times \mu(V_e) \rightarrow V_i \cup \{\text{ERROR}\} \quad \text{as}$$

$$\psi(\lambda(u), \mu(u)) = \begin{cases} a & \text{if there is the production } \lambda(u) \rightarrow a\mu(u)u \in F \\ \text{ERROR} & \text{otherwise} \end{cases}$$

$$(\text{it has been noted } f(A) = \{f(x) \mid x \in A\}).$$

Let $u_1 u_2 \dots u_n$ be the word sent by the channel. If no error appeared (consequently this is the code-word), the decoded sequence would be

$$\psi(\lambda(u_1), \mu(u_1)) \psi(\lambda(u_2), \mu(u_2)) \dots \psi(\lambda(u_n), \mu(u_n)) \dots$$

The moment an error appears, we have noticed that any wrong character could be replaced with an element of V_e . So, the error can be corrected (if we have the chance), or it is detected and would be corrected later.

All the decodification algorithms must clarify two cases, namely

- Finding out the position where the wrong character u_k appears.
- Finding out the correct character.

The first algorithm solves these problems on the basis of the following

Theorem: The tree-codification φ corrects every error in the hypothesis (2) if and only if $l \geq 3$.

Prof: "if": We suppose that $l = 2$. Then, there are two states

φ_β and φ_α with $\varphi_\alpha \Rightarrow a_1 \varphi_{\alpha_1} u_1 \Rightarrow a_1 a_2 \varphi_\beta u_2 u_1$ and

$$\varphi_\alpha \Rightarrow b_1 \varphi_{\alpha_2} u_3 \Rightarrow b_1 b_2 \varphi_\beta u_4 u_3 \text{ where } \varphi_{\alpha_1} \neq \varphi_{\alpha_2}$$

We notice that to the sequence $u_1 u_2 \dots u_n \dots$ could be attached uniquely the sequence of pairs

$$(3) \quad (\lambda(u_1), \mu(u_1)) (\lambda(u_2), \mu(u_2)) \dots (\lambda(u_n), \mu(u_n)) \dots$$

In our hypothesis there would be possible a sequence of the form

$$(\varphi_\gamma, \varphi_\alpha) (\varphi_\alpha, \varphi_{\alpha_1}) (\varphi_{\alpha_2}, \varphi_\beta) (\varphi_\beta, \varphi_\delta) \dots$$

which could be framed in (2) but at a certain moment it has a cut; hence one of the characters $\psi(\varphi_\alpha, \varphi_{\alpha_1})$ or $\psi(\varphi_{\alpha_2}, \varphi_\beta)$ is wrong.

However we cannot tell which exactly.

"only if": Let it be the sequence (3). It is noticed that if a character is received correctly or even wrongly received the error is not detected immediately, we have

$$\mu(u_{k-1}) = \lambda(u_k), \quad k = 1, 2, 3, \dots$$

In the hypothesis (2) sequence

$$(\lambda(u_{k-1}), \mu(u_{k-1})) (\lambda(u_k), \mu(u_k)) (\lambda(u_{k+1}), \mu(u_{k+1}))$$

let a cut be between the first two pairs, viz. $\mu(u_{k-1}) \neq \lambda(u_k)$.

We check immediately if $\mu(u_k) = \lambda(u_{k+1})$. If so, u_k has been wrongly received, because we are in the hypothesis (2).

Correction is made by replacing $(\mu(u_{k-1}), \lambda(u_{k+1}))$.

If $\mu(u_k) = \lambda(u_{k+1})$, we do not know then if the first or the second character has been wrongly received. In any case the elements $\lambda(u_{k-1})$ and $\mu(u_k)$ are known as equal with $\mu(u_{k-2})$ ($k > 2$, or if $k = 2$), $\lambda(u_{k+1})$ respectively. If $D(\lambda(u_{k-1})) \cap S(\mu(u_k))$ has more than one element, we do not know which character is wrong and what is its correction. In order to be sure, $D(\lambda(u_{k-1})) \cap S(\mu(u_k))$ should have one element at most, namely we have to be under the conditions of Proposition above; consequently $l \geq 3$.

On the basis of this theorem we give the following error detection and correction algorithm:

Algorithm 1:

1. $k = 1$
2. If $\lambda(u_1) \neq \varphi$ then $a_1 = \psi(\varphi, \lambda(u_2))$, else $a_1 = \psi(\lambda(u_1), \mu(u_1))$, $\lambda(u_2) = \mu(u_1)$.
3. $k := k + 1$; if $\mu(u_{k+2})$ undefined, STOP
4. If $\mu(u_k) = \lambda(u_{k+1})$ then $a_k = \psi(\lambda(u_k), \mu(u_k))$, go to 3.
5. If $\mu(u_{k+1}) \neq \lambda(u_{k+2})$ then $a_k = \psi(\lambda(u_k), \mu(u_k))$, $\lambda(u_{k+1}) = \mu(u_k)$, go to 3.
6. $C = D(\lambda(u_k)) \cap S(\mu(u_{k+1}))$
7. If $C = \emptyset$ then uncorrectable error, STOP
8. $\mu(u_k) \in C$, then $a_k = \psi(\lambda(u_k), \mu(u_k))$, go to 3.
9. $a_k = \psi(\lambda(u_k), \lambda(u_{k+1}))$, go to 3.

It is remarkable that in this algorithm it is not possible to correct the last symbol; indeed, this symbol appears as the application of a production of the form $\varphi_\alpha \rightarrow au$ where $\mu(u)$ has not been defined

Therefore, in order to correct this symbol it is preferable that it should be sent with a production of the form $\varphi_\alpha \rightarrow a\varphi_\beta u$ and in the end to be sent an arbitrary character v given by $\varphi_\beta \rightarrow bv$,

in the hypothesis (1). To this end we shall extend the predecessor and the successor functions: $P(A^\varphi) \rightarrow P(A^\varphi)$ as follows:

$$\begin{aligned} D^j(\varphi_\alpha) &= \{\varphi_\beta \mid \varphi_\beta \in D(\varphi_\gamma), \varphi_\gamma \in D^{j-1}(\varphi_\alpha)\} \quad D^0(\varphi_\alpha) = \{\varphi_\alpha\} \\ S^j(\varphi_\alpha) &= \{\varphi_\beta \mid \varphi_\beta \in S(\varphi_\gamma), \varphi_\gamma \in S^{j-1}(\varphi_\alpha)\} \quad S^0(\varphi_\alpha) = \{\varphi_\alpha\} \quad j \geq 1 \end{aligned}$$

We used the conventional denotation $D(A) = \{y \mid y \in D(x), x \in A\}$ and

$$S(A) = \{y \mid y \in S(x), x \in A\}$$

Now, we can give the following proposition:

Proposition: The tree-codification φ has the minimal constraint length l , if and only if for every $t < l$, $0 \leq m \leq t-1$. $\varphi_\alpha, \varphi_\beta \in A$, there is $|D^m(\varphi_\alpha) \cap S^{t-m-1}(\varphi_\beta)| < 1$

Proof: We suppose that $\varphi_{\gamma_1}, \varphi_{\gamma_2} \in D^m(\varphi_\alpha) \cap S^{t-m-1}(\varphi_\beta)$. This is equivalent with the fact that there are the derivations

$$\varphi_\alpha \xRightarrow{*} \alpha_1 \varphi_{\gamma_1} u_1 \mid \alpha_2 \varphi_{\gamma_2} u_2, \quad \varphi_{\gamma_1} \xRightarrow{*} \beta_1 \varphi_\beta v_1, \quad \varphi_{\gamma_2} \xRightarrow{*} \beta_2 \varphi_\beta v_2$$

with $l(\alpha_1) = l(\alpha_2) = m$, $l(\beta_1) = l(\beta_2) = t - m - 1$.

Hence will be possible the derivations $\varphi_\alpha \xRightarrow{*} \alpha_1 \beta_1 \varphi_\beta v_1 u_1$, $\varphi_\alpha \xRightarrow{*} \alpha_2 \beta_2 \varphi_\beta v_2 u_2$ with $\alpha_1 \beta_1 \neq \alpha_2 \beta_2$ and $l(\alpha_1 \beta_1) = l(\alpha_2 \beta_2) = t - 1$. Then the minimal constraint length will be $t - 1 < l - 1$, contradiction.

Based on this, we shall give the following algorithm:

Algorithm 2:

1. $k := 1$, $E = \emptyset$, $T = \{u \mid \varphi \rightarrow a \varphi_\alpha \quad u \in F\}$, $\mu(u_0) := \varphi$
2. $n := 0$, $i := 1$. If $\lambda(u_k) \neq \mu(u_{k+1})$, go to 6
3. If $\mu(u_k) \notin S^{i-1}(\lambda(u_{k+1}))$, $n := n + 1$.
4. $i := i + 1$. If $i \leq l - 1$, go, to 3.
5. If $n < \left\lceil \frac{l-1}{2} \right\rceil$, $a_k := \psi(\lambda(u_k), \mu(u_k))$, go to 8.
6. $E := E \cup \{u_k\}$. If $T - E = \emptyset$, uncorrected error, STOP.
7. On take $x \in T - E$, $u_k := x$, go to 2.
8. $E := \emptyset$, $k := k + 1$, $T := \{u \mid \mu(u_{k+1}) \rightarrow a \varphi_\alpha \quad u \in F\}$
9. If there is $u_k u_{k+1} \dots u_{k+l-1}$ go to 2, else STOP.

We can notice that in this algorithm, when we pass over a character, it is already decodified, as compared to the first algorithm where there was possible a coming back to the previous character and even its correction.

Theorem 2: Algorithm 2 detects and corrects every error in the hypothesis (2).

Proof: Let a sequence be of l successive characters $u_k \dots u_{k+l-1}$. All the characters till u_k have been correctly decoded and in T we have the possible values which could be taken by the k -th character received. If u_k is correct, then in the first $l-1$ positions we shall have at least $\left\lfloor \frac{l-1}{2} \right\rfloor + 1$ nodes of the path, because $\mu(u_k)$ is the root of the tree which generates the remained part of the code word. Namely $\mu(u_k) \in S^{i-1}(\lambda(u_{k+i}))$, $i = 1, 2, \dots, l-1$ is true for at least $\left\lfloor \frac{l-1}{2} \right\rfloor + 1$ values of i . Then $n < l - 1 - \left\lfloor \frac{l-1}{2} \right\rfloor - 1 < \left\lfloor \frac{l-1}{2} \right\rfloor$ and the decodification is achieved through step 5; u_k is eliminated, the whole word is moved to the left with one position and u_{k+i} is introduced to the right extremity.

If u_k is not correct, there are two more possibilities left:

a) $u_k \in T$ and after having gone through the steps 2, 6, 7 it is eliminated and we look for the correct character among the elements of T .

b). $u_k \in T$. Then, for at least $l - \left\lfloor \frac{l-1}{2} \right\rfloor$ elements, the step 3 is verified, so n increases till it passes over $\left\lfloor \frac{l-1}{2} \right\rfloor$. This one is based on the Proposition and the hypothesis (1). Indeed, if u_{k+p} would be correct and $\mu(u_k) \in S^{p-1}(\lambda(u_{k+p}))$, then

$|D(\mu(u_{k-1})) \cap S^{p-1}(\lambda(u_{k+p}))| > 1$ that it is in contradiction with the Proposition. With step 6, u_k is eliminated from the elements considered to be correct and we look for another element in the set T .

We note:

1). The algorithm 2 cannot correct the last l characters of the sequence received. As a matter of fact it is convenient to be used when l is large enough and the code-word sent sufficiently long.

2). In this algorithm n gives the number of errors which appeared in the sequence of the following, l characters received after an arbitrary character; T gives the set of characters V_e which could be received at an arbitrary moment. E consists of the set of characters which have violated their own conditions of appearing at that moment.

Example 3: Let us take the tree-code built on the basis of the tree-codification given in the first part of the paper. Let us suppose that the sequence xx_2y_3 is received. The code, having $l = 3$, could correct at most an error. When $i = 1$, $\mu(x) = \varphi_0$ and $\lambda(x_2) = \varphi_1$, hence $n = 1$. Then, for $i = 2$, because $S(\varphi_{10}) = \{\varphi_1, \varphi_{10}\}$ it results $n = 2$. The algorithm goes to the step 6 and makes $E = \{x\}$, then it takes another element of $T - E$, namely y . This will be the correct symbol, consequently the correct sequence is yx_2y_3 . y is decoded in 1 and the algorithm goes to the following sequence of three characters.

If xy_2y_1 is received, the first character is correctly decoded although it signals $n = 1$. As for the second character we have $\mu(x) = \varphi_0$, $\lambda(x_2) = \varphi_1$ and step 2 sends to 6 which proves x_2 is wrong and replaces it with another element of T .

x_1 is found and verifies the algorithm.

4. The algorithm of decodification in variable steps.

This is a simple method of decodification, similar to the methods used in the Algebraic — bloc Codes. Decodification and error correction is achieved on sequences variable as length, and not character by character. First, let us give some definitions and properties useful in the algorithm writing.

Definition: Let $\alpha, \beta \in V_e^*$ be the two words with $l(\alpha) = l(\beta) < \infty$. We call $d_H(\alpha, \beta)$ — Hamming's metric — as the number of positions in which can differ α and β .

For example, if $\alpha = xyx_1y_3$, $\beta = x_1yx_1y$, we have $d_H(\alpha, \beta) = 2$, because they differ in the positions 1 and 4.

Let $G = (A^\varphi, V_i, V_e, \varphi, F)$ be a C-grammar of a tree-codification φ achievable in it. For each $\varphi_\alpha \in A^\varphi$, there is the set

$$H_{\varphi_\alpha} = \{w \mid w = \varphi_\alpha(\beta), l(\beta) = l\}$$

that is, H_{φ_α} is the set of all the words of length l from $\partial_\alpha L_e$ ($/2/$)

Proposition A: If $\beta_1, \beta_2 \in H_{\varphi_\alpha}$, $d_H(\beta_1, \beta_2) = k$, then $\beta_1 = \gamma\delta_1$, $\beta_2 = \gamma\delta_2$, with $l(\gamma) = l - k$.

Proof: We suppose $l(\gamma) = l - k$. Then there is at least one character $x \in V_e$ such that $\beta_1 = \gamma w_1 x u_1$, $\beta_2 = \gamma w_2 x u_2$ with $l(w_1) = l(w_2)$, $w_1 \neq w_2$. Let $\varphi_\beta \rightarrow a\varphi_\alpha x$ be the production in G which introduces x . Then there will be the derivations $\varphi_\alpha \Rightarrow \alpha' \beta_\gamma w_1 \mid \alpha'' \beta_\gamma w_2$ with $l(\alpha') = l(\alpha'') < l$ but $\alpha' \neq \alpha''$ that contradicts the definitor of the minimum constraint length l . Hence, if two words of H_{φ_α} vary in k positions, these are necessarily the last k positions; consequently $l(\gamma) = l - k$.

Proposition B: Let it be $\gamma \in V_e^*$, $l(\gamma) = l$. If there is a word α_1 in H_{φ_α} with $d_H(\alpha_1, \gamma) < \left\lceil \frac{l-1}{2} \right\rceil$, then there is $\beta_1, \beta_2, \dots, \beta_n \in H_{\varphi_\alpha}$ ($n > 1$) with the following properties:

a) $d_H(\gamma, \beta_i) < d_H(\gamma, \beta)$ for every $\beta \in H_{\varphi_\alpha}$

b) $d_H(\beta_i, \beta_j) < l-1$, $i, j = 1, 2, \dots, n$

Proof: It is obvious that these elements are in H_{φ_α} ; α_1 is an exemple.

We are interested in finding out how many of these are there.

If $n = 1$, β is unique and $d_H(\beta, \beta) = 0 < l-1$

We suppose that there are more words β which can achieve this minimum.

If there is β_1, β_2 with $d_H(\beta_1, \beta_2) = l$, then β_1 and β_2 have no common character. Consequently a word γ situated at an equal distance to β_1 and β_2

will differ from β_1 and β_2 through an equal number of characters.

$$\text{Hence } d_H(\gamma, \beta_i) > \left\lfloor \frac{l}{2} \right\rfloor + 1 > \left\lfloor \frac{l-1}{2} \right\rfloor$$

This contradicts the fact that β_1 and β_2 are elements of $H_{\varphi\alpha}$ placed at the minimum distance to γ .

We denote $d_H(\alpha, A) = \min \{d_H(\alpha, \beta) | \beta \in A\}$

We can give now the algorithm of decodification in variable steps based on Hamming's metric:

Algorithm 3:

1. $k := 1, \mu(\hat{u}_0) := \varphi$
2. Let it $\alpha = u_k u_{k-1} \dots u_{k+l-1}, H := H_{\mu(\hat{u}_{k-1})}$
3. If $d_H(\alpha, H) > \left\lfloor \frac{l-1}{2} \right\rfloor$, uncorrected error, STOP.
4. Let it be $\beta_1, \beta_2, \dots, \beta_n \in H$ given by the Proposition. B. With the Proposition A, there is p and $\gamma = v_1 v_2 \dots v_p \in V_e^*$ with $\beta_i = \gamma \delta^i$
 $i = 1, 2, \dots, n$.
5. $\hat{u}_{k+i-1} := v_i, a_{k+i-1} := \psi(\lambda(u_{k+i-1}), \mu(\hat{u}_{k+i-1}))$ for all $i, 1 < i < \leq \min \left\{ \left\lfloor \frac{l}{2} \right\rfloor, p \right\}$.
6. $k := k + \min \left\{ \left\lfloor \frac{l}{2} \right\rfloor, p \right\}$. If there is $u_k \dots u_{k+l-1}$, go to 2, else

STOP.

We use here notation u_k for the character to be analysed and with $\hat{u}_k$ for the decoded character.

Theorem: The algorithm of decodification in variable steps decodifies and corrects errors for every code-word which verifies (1).

Proof: The set H contains all the possible sequences of length l which could appear in the code-word at a certain moment. If there appear more than $\left\lfloor \frac{l-1}{2} \right\rfloor$ errors, then the algorithm is either stopped at step 3 or it

fails the right decodification. We suppose that there appeared $t < \left\lfloor \frac{l-1}{2} \right\rfloor$

errors. There will be at least one word β with $d_H(\alpha, \beta) = t$ in H . The more errors appear at the beginning of the sequence, the more increases the possibility of decodification more characters at one step; this is like that because the more correct character we have in the last positions, the more decreases the number of different words $\beta_1, \beta_2, \dots, \beta_n$ which could be possible decodification of α . However, no matter what the correct sequence would be what was to be surely decodified is but the common root of all those.

But we cannot make attempts beyond the half of the word. There is the possibility in there that all the errors be gathered in the second part of the word and so lead it to another code-sequence. Property b) of Proposition B assure that at least one character is decodified at every step.

Example 4: Let it be the convolutional code ([3]) given by the next tree-structure shown in figure 3.

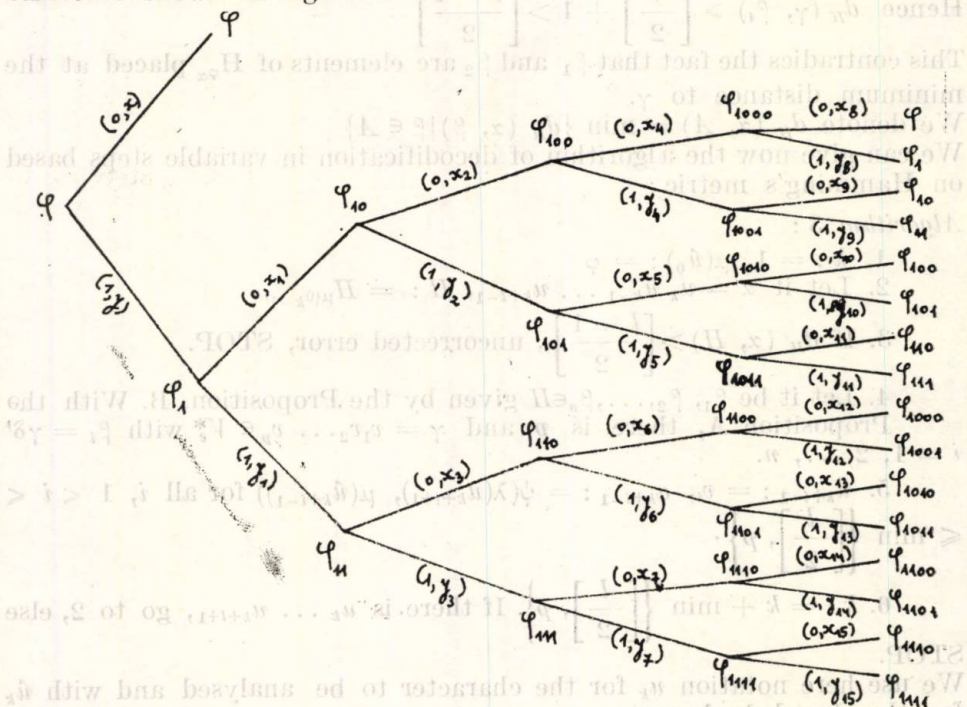


Fig. 3.

We have $l = 5$ in this code. We suppose that the sequence $y_1 x_1 x_3 y_6 x_8 x_{11} \dots$ has been received.

We take $\alpha = y_1 x_1 x_3 y_6 x_8$ and $H = H_\varphi$ as the first decodification.

We have $d_H(\alpha, H) = 2$ and three words $\beta_1 = y_1 x_1 x_3 y_6 x_8$, $\beta_2 = y_1 x_1 x_3 y_6 x_{13}$, $\beta_3 = y_1 x_1 x_3 y_6 y_{13}$; Hence $p = 1$, $\gamma = y$. The first character y is decodified and we pass to the following sequence $\alpha = x_1 x_3 y_6 x_8 x_{11}$.

Here $H = H_{\varphi_1}$ and $d_H(\alpha, H) = 2$; consequently there are two errors in this sequence. We obtain the word β_1 , $y_1 x_1 x_3 y_6 x_{13} x_{11}$. Hence $p = 5$.

The sequence $y_1 x_3$ will be decodified (we also notice correction of x_1 with y_1) and we pass to the following sequence.

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Mathematical Faculty
University of Bucharest

ON THE UNIQUENESS OF THE SOLUTION OF A SYSTEM OF CONTEXT-FREE EQUATIONS

VIRGIL EMIL CĂZĂNESCU

A sufficient condition is given for the uniqueness of the solution of a system of context-free equations. As the ideal systems of regular equations verify this condition, our theorem is a generalization of the theorem on the uniqueness of the solution of an ideal system of regular equations (C.C. Elgot and J.B. Wright).

Notations are essentially the same as in [3].

Let $\Delta = \{n_i\}_{i \in I}$ be a family of nonnegative integers and M a set; we suppose I and M to be disjoint. For $m \in M$ we set $n_m = 0$.

Let ω be the set of nonnegative integers, ω^* the free monoid generated by ω and ε its identity.

Let further $CT_\Delta(M)$ be the set of all subsets a of $\omega^* \times (I \cup M)$ which satisfy the properties

1) $(\alpha, x) \in a$ and $(\alpha, y) \in a$ imply $x = y$

2) if $\alpha \in \omega^*$, $n \in \omega$, $(\alpha n, x) \in a$, then there exists $i \in I$ such that $(\alpha, i) \in a$ and $n < n_i$.

We shall identify $m \in M$ with $\{(\varepsilon, m)\}$; thus we shall assume $M \subset CT_\Delta(M)$.

If $i \in I$ and $\{a_0, a_1, \dots, a_{n_i-1}\} \subset CT_\Delta(M)$ then $f_i(a_0, a_1, \dots, a_{n_i-1}) = \{(\varepsilon, i)\} \cup \bigcup_{k=0}^{n_i-1} \{(k\alpha, j) \mid (\alpha, j) \in a_j\}$ defines an n_i -ary operation on $CT_\Delta(M)$.

$\mathcal{A} = (A, \{g_i\}_{i \in I}, <, 0)$ is said to be an ω -continuous algebra of type Δ if:

1) $<$ is a partial order on A , 0 is the least element, and every increasing sequence in A has a least upper bound ($\vee$)

2) for every $i \in I$, $g_i: A^{n_i} \rightarrow A$ is ω -continuous, i.e.,

$$g_i(\bigvee_n x^n) = \bigvee_n g_i(x^n)$$

for every increasing sequence $x^n \in A^{n_i}$.

$(CT_\Delta(M), \{f_i\}_{i \in I}, \subseteq, \emptyset)$ is an ω -continuous algebra of type Δ , freely generated by M , that is, for every ω -continuous algebra $\mathcal{A}$ of type Δ , every function $f: M \rightarrow A$ can be uniquely extended to a homomorphism $\bar{f}: CT_\Delta(M) \rightarrow A$ of ω -continuous algebras

$$[\bar{f}(f_i(a_0, a_1, \dots, a_{n_i-1})) = g_i(\bar{f}(a_0), \bar{f}(a_1), \dots, \bar{f}(a_{n_i-1}))].$$

$\bar{f}(\emptyset) = 0$ and $\bar{f}$ is ω -continuous].

In order to show that a property P holds for every $a \in CT_{\Delta}(M)$, it suffices to prove that:

- 1) $\emptyset$ has property P ;
- 2) every $m \in M$ has property P ;
- 3) if $i \in I$ and $a_0, a_1, \dots, a_{n_i-1}$ have property P , then $f_i(a_0, a_1, \dots, a_{n_i-1})$ has also property P ;
- 4) if $\{a_n\}_{n \in \omega}$ is an increasing sequence of elements with property P , then $a = \bigvee_n a_n$ has also property P .

The reasoning based on this remark is called *continuous induction*.

Let $n = \{0, 1, \dots, n-1\} \in \omega$, $\mathbf{a} \in CT_{\Delta}(n)$ and $\mathcal{A}$ an ω -continuous algebra. We shall denote by $a_{\mathcal{A}}$ the n -ary operation on A induced by $\mathbf{a}$ which is defined by $a_{\mathcal{A}}(f) = \bar{f}(\mathbf{a})$ for every $f: n \rightarrow A$. Using continuous induction one can show that $a_{\mathcal{A}}$ is ω -continuous.

Theorem. If $n \in \omega$, $g \in CT_{\Delta}(n)$ and $\{a_0, a_1, \dots, a_{n-1}\} \subseteq CT_{\Delta}(M)$, then

$$g_{CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}) = (g \setminus (\omega^* \times n)) \cup \{(\alpha\beta, i) / (\alpha, k) \in g, k \in n \text{ and } (\beta, i) \in a_k\}.$$

Proof by continuous induction.

1) $g = \emptyset$. It follows from the definition of the induced operation that $g_{CT_{\Delta}}(a_0, a_1, \dots, a_{n-1}) = \emptyset$. therefore the desired equality is verified.

2) $g \in n$. It follows from the definition that $g_{CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}) = a_g$, hence the equality holds in this case as well.

3) $g = f(g_0, g_1, \dots, g_{n_i-1})$ with $i \in I$ and for every $k \in n_i$, the element $g_k \in CT_{\Delta}(n)$ verifies the desired equality. $g_{CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}) =$

$$= f_i(g_{0, CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}), \dots, g_{n_i-1, CT_{\Delta}(M)}(a_0, \dots, a_{n-1})) =$$

$$= \{(\varepsilon, i)\} \cup \bigcup_{k=0}^{n_i-1} \{(k\alpha, j) / (\alpha, j) \in g_{k, CT_{\Delta}(M)}(a_0, \dots, a_{n-1})\} =$$

$$= \{(\varepsilon, i)\} \cup \bigcup_{k=0}^{n_i-1} \{(k\alpha, j) / (\alpha, j) \in g_k, j \notin n\} \cup$$

$$\cup \bigcup_{k=0}^{n_i-1} \{(k\alpha, \beta, u) / (\alpha, j) \in g_k, j \in n, (\beta, u) \in a_j\} =$$

$$= (g \setminus (\omega^* \times n)) \cup \{(\alpha\beta, u) / (\alpha, k) \in g, k \in n \text{ and } (\beta, u) \in a_k\}$$

4) Let $\{g_r\}_{r \in \omega}$ be an increasing sequence from $CT_{\Delta}(n)$ such that all g_r verify the equality and let $g = \bigcup_{r \in \omega} g_r$. Then

$$g_{CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}) = \bigcup_{r \in \omega} g_{r, CT_{\Delta}(M)}(a_0, a_1, \dots, a_{n-1}) =$$

$$= \bigcup_{r \in \omega} [(g_r \setminus (\omega^* \times n)) \cup \{(\alpha\beta, u) / (\alpha, k) \in g_r, k \in n, (\beta, u) \in a_k\}] =$$

$$= ((\bigcup_{r \in \omega} g_r) \setminus (\omega^* \times n)) \cup \{(\alpha\beta, u) / (\alpha, k) \in \bigcup_{r \in \omega} g_r, k \in n, (\beta, u) \in a_k\}$$

In the sequel we assume that $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$ and we put $\Delta_1 = \{n_i\}_{i \in I_1}$.

By a *system of context-free equations* we mean a mapping $E : I_2 \rightarrow CT_\Delta(\omega)$ such that $E(i) \in CT_\Delta(n_i)$ for every $i \in I_2$. Although in the literature it is assumed that $E(i)$ is finite, this hypothesis is not necessary to the theorem we will prove.

The solutions of the context-free system E will be defined as elements of the set $A = \Pi \{CT_{\Delta_1}(n_i) / i \in I_2\}$, having a certain property. To be specific, using the product ordering of A we note that A has a least element and every increasing sequence in A has a least upper bound. Let $s = \{s_j\}_{j \in I_2}$ be an element of A . For every $i \in I_2$, we denote by

$$\lambda_s^i : CT_\Delta(n_i) \rightarrow (CT_{\Delta_1}(n_i), \{f_j\}_{j \in I_1}, \{s_{j, CT_{\Delta_1}(n_i)}\}_{j \in I_2}, \subseteq, \emptyset)$$

the unique homomorphism of ω — continuous algebras of type Δ for which $(\forall k \in n_i) (\lambda_s^i(k) = k)$. Let $\varphi : A \rightarrow A$ be the function defined by $\varphi(s)(i) = \lambda_s^i(E(i))$. The family s is said to be a *solution* of E provided $\varphi(s) = s$. As φ is ω — continuous, it follows that E has a least solution.

Definition. The system E of context-free equations is said to be *ideal* if

$$(1) \quad (\forall i \in I_2) (E(i) \notin n_i)$$

and

(2) there is no function $r : \omega \rightarrow I_2$ such that

$$(\forall k \in \omega) (\varepsilon, r(k+1)) \in E(r(k)).$$

Condition (2) enables us to prove by induction properties P concerning the set I_2 . This type of induction, which we shall call *tree-like*, consists of the following steps:

1) One shows that P is true for those $i \in I_2$ for which

$$I_2(\varepsilon, j \in (\forall j) j \notin E(i)).$$

2) If $i \in I_2$ has property P , $j \in I_2$ and $(\varepsilon, i) \in E(j)$, then j has property P .

Theorem. The solution of an ideal system of context-free equations is unique.

Comments. Condition (1), which is introduced just for the sake of convenience, might be omitted, because for every $i \in I_2$ such that $E(i) \in n$ the i -th equation can be eliminated from the system.

If $n_i = 0$ for every $i \in I_2$, then E is said to be a system of *regular* equations. If, moreover

$$(\forall i \in I_2) (\forall j \in I_2) (E(i) \neq \{(\varepsilon, j)\}).$$

then E is said to be an ideal system of regular equations. These definitions are obviously equivalent to those used in the literature. It is easy to see



that such a system is an ideal system of context-free equations, therefore the theorem to be proved, as well as other theorems about systems of context-free equations, is a generalization of the corresponding theorems on systems of regular equations.

Proof. Let $s \in A$ be a solution of E . We first show by tree-like induction that $s_i \notin n_i$ for every $i \in I_2$.

a) Let $i \in I_2$ such that $(\forall j \in I_2) (\varepsilon, j) \notin E(i)$. As $E(i) \notin n_i$, it follows that $E(i) = \emptyset$ or $E(i) = f_j(a_0, \dots, a_{n_j-1})$ for some $j \in I_1$ and some $a_k \in CT_\Delta(n_i)$ for $k \in n_j$. If $E(i) = \emptyset$ then $s_i = \lambda_s^i(\emptyset) = \emptyset$, hence $s_i \notin n_i$. Otherwise $s_i = \lambda_s^i(E(i)) = f_j(\lambda_s^i(a_0), \dots, \lambda_s^i(a_{n_j-1})) \notin n_i$.

b) Suppose $i \in I_2$, $j \in I_2$, $s_i \notin n_i$ and $(\varepsilon, i) \in E(j)$; we shall prove that $s_j \notin n_j$. As $(\varepsilon, i) \in E(j)$ it follows that $E(j) = f_i(a_0, a_1, \dots, a_{n_i-1})$ with $a_k \in CT_\Delta(n_j)$ for $k \in n_i$, therefore

$$\begin{aligned} s_j &= \lambda_s^j(E(j)) = s_{i, CT_{\Delta_1}(n_j)}(\lambda_s^j(a_0), \dots, \lambda_s^j(a_{n_i-1})) = \\ &= (s_i / (\omega^* \times n_i)) \cup \{(\alpha\beta, v) / (\beta, v) \in \lambda_s^j(a_u), (\alpha, u) \in s_i, u \in n_i\}. \end{aligned}$$

Now the assumption $s_j \in n_j$ implies the existence of $u \in n_i$, $(\alpha, u) \in s_i$ and $(\beta, v) \in \lambda_s^j(a_u)$ such that $\alpha\beta = \varepsilon$ and $v = s_j$, hence $(\varepsilon, u) \in s_i$, that is $s_i = \{(\varepsilon, u)\} = u \in n_i$, which contradicts the induction hypothesis.

Let further s and t be two solutions of E . We shall prove by induction on n that

$$(\forall n \in \omega) (\forall i \in I_2) (\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow s_i(\alpha) = t_i(\alpha)),$$

where $|\alpha|$ denotes the length of α and we adopt the convention that the equality $s_i(\alpha) = t_i(\alpha)$ implies in particular that α belongs to the domain of s_i if and only if it belongs to the domain of t_i .

Let $n \in \omega$ fulfil

$$(\forall i \in I_2) (\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow s_i(\alpha) = t_i(\alpha)).$$

We shall prove that

$$(\forall \alpha \in \omega^*) (|\alpha| = n \Rightarrow s_i(\alpha) = t_i(\alpha)).$$

A) Let us show by continuous induction that

$$(\forall a \in CT_\Delta(n_i)) (\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow \lambda_s^i(a)(\alpha) = \lambda_t^i(a)(\alpha)).$$

If $a = \emptyset$ then $\lambda_s^i(a) = \emptyset = \lambda_t^i(a)$.

If $a \in n_i$ then $\lambda_s^i(a) = a = \lambda_t^i(a)$.

Let $a = f_j(a_0, \dots, a_{n_j-1})$ where $j \in I$ and the relations $a_k \in CT_\Delta(n_i)$ and

$$(\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow \lambda_s^i(a_k)(\alpha) = \lambda_t^i(a_k)(\alpha))$$

hold for each $k \in n_j$.

If $j \in I_1$ then

$$\lambda_s^i(a) = f_j(\lambda_s^{i'}(a_0), \dots, \lambda_s^{i'}(a_{n_j-1})) = \{(\varepsilon, j)\} \cup \bigcup_{k=0}^{n_j-1} \{(k\alpha, u) / (\alpha, u) \in \lambda_s^i(a_k)\}$$

and a similar equality holds for $\lambda_t^i(a)$, therefore

$$(\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow \lambda_s^i(a)(\alpha) = \lambda_t^i(a)(\alpha)).$$

If $j \in I_2$ then

$$\lambda_s^i(a) = s_{j, CT_\Delta(n_i)}(\lambda_s^i(a_0), \dots, \lambda_s^i(a_{n_j-1})) =$$

$$= (s_j \setminus (\omega^* \times n_j)) \cup \{(\alpha\beta, u) / (\beta, u) \in \lambda_s^i(a_k), (\alpha, k) \in s_j \text{ and } k \in n_j\}$$

and a similar equality holds for $\lambda_t^i(a)$, therefore

$$(\forall \alpha \in \omega^*) (|\alpha| < n \Rightarrow \lambda_s^i(a)(\alpha) = \lambda_t^i(a)(\alpha)).$$

B) We prove by tree-like induction that

$$(\forall i \in I_2) (\forall \alpha \in \omega^*) (|\alpha| = n \Rightarrow s_i(\alpha) = t_i(\alpha)).$$

a) If $E(i) = \emptyset$ then $s_i = t_i$.

If $E(i) = f_j(a_0, a_1, \dots, a_{n_j-1})$ with $j \in I_1$ and

$a_k \in CT_\Delta(n_i)$ for $k \in n_j$ then

$$s_i = \{(\varepsilon, j)\} \cup \bigcup_{k=0}^{n_j-1} \{(k\alpha, u) / (\alpha, u) \in \lambda_s^i(a_k)\}$$

and a similar relation holds for t_i , therefore

$$(\forall \alpha \in \omega^*) (|\alpha| = n \Rightarrow s_i(\alpha) = t_i(\alpha)).$$

b) Suppose $i \in I_2, j \in I_2, (\varepsilon, i) \in E(j)$ and

$$(\forall \alpha \in \omega^*) (|\alpha| = n \Rightarrow s_i(\alpha) = t_i(\alpha)). \text{ As}$$

$E(j) = f_i(a_0, \dots, a_{n_j-1})$, where $a_k \in CT_\Delta(n_j)$ for $k \in n_i$, it follows that

$$s_j = (s_i \setminus (\omega^* \times n_i)) \cup \{(\alpha\beta, u) / (\beta, u) \in \lambda_s^j(a_k), (\alpha, k) \in s_i, k \in n_i\}.$$

As $s_i \notin n_i$, it follows that

$$(k \in n_i \text{ and } (\alpha, k) \in s_i) \Rightarrow \alpha \neq \varepsilon.$$

It follows, from the similar relation for t_j , that

$$(\forall \alpha \in \omega^*) (|\alpha| = n \Rightarrow s_j(\alpha) = t_j(\alpha)).$$

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Mathematical Faculty
University of Bucharest

A WEIERSTRASS-TYPE THEOREM FOR VECTOR RANDOM FUNCTIONS.

GHEORGHE CENUȘĂ

Abstract

In this paper the results obtained in [1] and [4] are generalized for vector valued random functions.

1. Preliminary results.

Let $\{\Omega, \mathcal{H}, P\}$ be a complete probability space and C^n the complex vector space. We denote by X the set of all random vectors $x: \Omega \rightarrow C^n$. With each $x \in X$ we associate the equivalence class $\tilde{x}$ induced by the relation:

$$x \equiv y, y \in X \Leftrightarrow P(\{\omega: x(\omega) \neq y(\omega)\}) = 0$$

We denote by $\tilde{X}$ the set of all equivalence classes.

Using [2] and [3] the mapping

$$\tilde{x} \rightarrow \|\tilde{x}\|_P = \inf_{\alpha > 0} \{\alpha + P(\{\omega: \|x(\omega)\| > \alpha\})\}, \quad x \in \tilde{x}$$

is introduced, where $\|x(\cdot)\|$ is the usual norm on C^n .

Obviously, the value $\|\tilde{x}\|_P$ does not depend on the chosen representative x and $0 \leq \|\tilde{x}\|_P \leq 1$. From [3] it is known that.

1. $\|-\tilde{x}\|_P = \|\tilde{x}\|_P$
2. $\|\tilde{x} + \tilde{y}\|_P \leq \|\tilde{x}\|_P + \|\tilde{y}\|_P$ for any $\tilde{x}, \tilde{y} \in \tilde{X}$
3. $\|\tilde{x}\|_P = 0 \Leftrightarrow \tilde{x} = 0$
4. If $x \in \tilde{x}$ and $P(\{\omega: \|x(\omega)\| > \eta\}) < \frac{\varepsilon}{2}$ with $0 < \eta \leq \frac{1}{2} \varepsilon$ then $\|\tilde{x}\|_P < \varepsilon$
5. If $\|\tilde{x}\|_P < \varepsilon$ then $P(\{\omega: \|x(\omega)\| > \varepsilon\}) < \varepsilon$ for any $x \in \tilde{x}$.

Denoting $\rho(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\|_P$ it follows that ρ is a metric on $\tilde{X}$ and the space $(\tilde{X}, \rho)$ is a complete metric space.

For each $x \in X$ we consider $\|x\|_P = \|\tilde{x}\|_P$. Then setting $\rho(x, y) = \|x - y\|_P$ it follows that ρ is a semimetric on X and (X, ρ) is a Fréchet space [2].

2. Stochastic approximation of vector random functions.

Consider the random function $f: [0,1] \rightarrow X$ and for any $n, m \in N$

$$\varphi_{m,n}: [0,1] \rightarrow R,$$

$$\varphi_{m,n}(t) = \begin{cases} 0 & t \in \left[0, \frac{m-1}{n}\right) \\ nt - m + 1 & t \in \left[\frac{m-1}{n}, \frac{m}{n}\right] \\ -nt + m + 1 & t \in \left(\frac{m}{n}, \frac{m+1}{n}\right] \\ 0 & t \in \left(\frac{m+1}{n}, 1\right], \quad m \leq n \end{cases}$$

It is known that the functions $\varphi_{m,n}$ are continuous.

Let us denote: $f_n(t, \omega) = \sum_{m=0}^n f\left(\frac{m}{n}, \omega\right) \varphi_{m,n}(t)$, $n \in N$

In this way we get a sequence of vector random functions $(f_n)_{n \in N}$

Theorem 1: If the vector random function f is ρ continuous on $[0,1]$, then the sequence of random functions $(f_n)_{n \in N}$ is uniformly ρ -convergent to the vector random function f .

Proof: Let $\varepsilon > 0$. There exists a natural number $k(\varepsilon)$ such that $\rho(f(t'), f(t'')) < \frac{\varepsilon}{4}$ for any $t', t'' \in [0,1]$ satisfying $|t' - t''| < \frac{1}{k(\varepsilon)}$ (the vector

random function f is uniformly ρ -continuous on $[0,1]$),

Using the property 5) from preliminary results we get that

$$P\left(\left\{\omega: \|f(t', \omega) - f(t'', \omega)\| > \frac{\varepsilon}{4}\right\}\right) < \frac{\varepsilon}{4}$$

Let $s \in [0,1]$ and $n > k(\varepsilon)$. There exists h , $0 \leq h \leq n$ such that $\frac{h-1}{n} < s \leq \frac{h}{n}$

Then $s = \frac{h-u}{n}$ where $0 \leq u < 1$ and $f_n(s, \omega) = u f\left(\frac{h-1}{n}, \omega\right) + (1-u) f\left(\frac{h}{n}, \omega\right)$.

$$\begin{aligned}
 \text{We have: } & \left\{ \omega : \|f(s, \omega) - f_n(s, \omega)\| > \frac{\varepsilon}{2} \right\} \subset \\
 & \left\{ \omega : u \left\| f(s, \omega) - f\left(\frac{h-1}{n}, \omega\right) \right\| > \frac{\varepsilon}{4} \right\} \cup \left\{ \omega : (1-u) \left\| f(s, \omega) - \right. \right. \\
 & \left. \left. - f\left(\frac{h}{n}, \omega\right) \right\| > \frac{\varepsilon}{4} \right\} \subset \left\{ \omega : \left\| f(s, \omega) - f\left(\frac{h}{n}, \omega\right) \right\| > \frac{\varepsilon}{4} \right\} \cup \\
 & \cup \left\{ \omega : \left\| f(s, \omega) - f\left(\frac{h-1}{n}, \omega\right) \right\| > \frac{\varepsilon}{4} \right\} \text{ and hence.}
 \end{aligned}$$

$$\begin{aligned}
 P\left(\left\{ \omega : \|f(s, \omega) - f_n(s, \omega)\| > \frac{\varepsilon}{2} \right\}\right) & \leq P\left(\left\{ \omega : \left\| f(s, \omega) - f\left(\frac{h}{n}, \omega\right) \right\| > \right. \right. \\
 & \left. \left. > \frac{\varepsilon}{4} \right\}\right) + P\left(\left\{ \omega : \left\| f(s, \omega) - f\left(\frac{h-1}{n}, \omega\right) \right\| > \frac{\varepsilon}{4} \right\}\right) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
 \end{aligned}$$

The property 4) preliminary results implies that for any $n > k(\varepsilon)$, $\rho(f(s), f_n(s)) < \varepsilon$. Since $k(\varepsilon)$ does not depend on s , the sequence of random functions $(f_n)_{n \in \mathbb{N}}$ is uniformly ρ -convergent to the vector random function f .

Theorem 2: Let $f: [0, 1] \rightarrow X$ be a random function with the properties:

- 1) f is ρ -continuous
- 2) $\sup_{t \in [0, 1]} \|f(t, \omega)\| \leq M < \infty$ for any $\omega \in \Omega$

Then there exists a sequence of random polynomials uniformly ρ -convergent to the random function f .

First proof: Let the sequence of random functions given by

$$f_n(t, \omega) = \sum_{m=0}^n f\left(\frac{m}{n}, \omega\right) \varphi_{m,n}(t)$$

and let $\varepsilon > 0$. There exists a natural number $N(\varepsilon)$ such that for any $n > N(\varepsilon)$ we have $\rho(f(t), f_n(t)) < \frac{\varepsilon}{4}$ whatever

$t \in [0, 1]$. Then, $P\left(\left\{ \omega : \|f(t, \omega) - f_n(t, \omega)\| > \frac{\varepsilon}{4} \right\}\right) < \frac{\varepsilon}{4}$ for any $t \in [0, 1]$ and any $n > N(\varepsilon)$. But for any fixed $n > N$ there exists a polynomial $P_{k(m,n)}$ such that,

$$|\varphi_{m,n}(t) - P_{k(m,n)}(t)| < \frac{\varepsilon}{4M(n+1)} \text{ for any } t \in [0, 1]$$

Let the sequence of polynomials $P_n(t, \omega) = \sum_{m=0}^n f\left(\frac{m}{n}, \omega\right) P_{k(m,n)}(t)$. It follows that $\|f(t, \omega) - P_n(t, \omega)\| \leq \|f(t, \omega) - f_n(t, \omega)\| + \|P_n(t, \omega) - f_n(t, \omega)\| \leq \|f(t, \omega) - f_n(t, \omega)\| + M \sum_{m=0}^n |\varphi_{m,n}(t) - P_{k(m,n)}(t)| < < \|f(t, \omega) - f_n(t, \omega)\| + \frac{\varepsilon}{4}$. Thus, $\left\{\omega : \|f(t, \omega) - P_n(t, \omega)\| > \frac{\varepsilon}{2}\right\} \subset \left\{\omega : \|f(t, \omega) - f_n(t, \omega)\| > \frac{\varepsilon}{4}\right\}$ and for any $n \geq N(\varepsilon)$, we have $P\left(\left\{\omega : \|f(t, \omega) - P_n(t, \omega)\| > \frac{\varepsilon}{2}\right\}\right) < \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$. It follows that if $n \geq N(\varepsilon)$, $\rho(f(t), P_n(t)) < \varepsilon$ for any $t \in [0, 1]$.

Second proof: Theorem 2 may be proved directly using the classical method. Indeed, consider the random polynomials

$$P_n(t, \omega) = \sum_{k=0}^n f\left(\frac{k}{n}, \omega\right) C_n^k t^k (1-t)^{n-k}$$

and let $\varepsilon > 0$. There exist $\delta(\varepsilon) > 0$ such that $\rho(f(t'), f(t'')) < \frac{\varepsilon}{4}$ for any $t', t'' \in [0, 1]$ satisfying $|t' - t''| < \delta(\varepsilon)$. Let $N(\varepsilon) \in \mathbb{N}$ be such that $\frac{M}{2n\delta^2(\varepsilon)} < \frac{\varepsilon}{4}$ for any $n \geq N(\varepsilon)$. We denote by I' the set of those k satisfying $\left|t - \frac{k}{n}\right| < \delta(\varepsilon)$ and by I'' the set of those k for which $\left|t - \frac{k}{n}\right| \geq \delta(\varepsilon)$ for any $t \in [0, 1]$. Then, $\|f(t, \omega) - P_n(t, \omega)\| \leq \sum_{k \in I'} C_n^k t^k (1-t)^{n-k} \|f(t, \omega) - f\left(\frac{k}{n}, \omega\right)\| + \sum_{k \in I''} C_n^k t^k (1-t)^{n-k} \left\|f(t, \omega) - f\left(\frac{k}{n}, \omega\right)\right\|$. Since

$$nt(1-t) = \sum_{k=0}^n (nt - k)^2 C_n^k t^k (1-t)^{n-k},$$

it follows that, $\sum_{k \in I'} C_n^k t^k (1-t)^{n-k} \left\|f(t, \omega) - f\left(\frac{k}{n}, \omega\right)\right\| \leq \frac{M}{2n\delta^2(\varepsilon)} < \frac{\varepsilon}{4}$, for any $n \geq N(\varepsilon)$. We have:

$$\left\{\omega : \|f(t, \omega) - P_n(t, \omega)\| > \frac{\varepsilon}{2}\right\} \subset \left\{\omega : \left\|f(t, \omega) - f\left(\frac{k}{n}, \omega\right)\right\| > \frac{\varepsilon}{4}\right\}$$

and $P\left(\left\{\omega : \|f(t, \omega) - P_n(t, \omega)\| > \frac{\varepsilon}{2}\right\}\right) < \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$ and hence $\rho(f(t), P_n(t)) < \varepsilon$ for any $t \in [0, 1]$, provided $n \geq N(\varepsilon)$.

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1°. The vibration of a moving road vehicle produce a lot of unpleasant effects as : failure due to random loading, diminution of safety in operation, discomfort of the transport, etc. An indicator of vibration was chosen for each of these effects and the problem was to minimize them in given conditions through the adequate choice of some parameters as elastic and damping characteristics, asymmetry coefficients or coupling coefficients. Examples of such indicators are the mean square acceleration of the sprung mass which indicates the confort, the dynamic force between the wheel and the road which is the indicator of the safety in operation.

In the following lines, the effect of the coupling coefficients on the mean square acceleration will be analyzed, to see how these parameters can be used in the road vehicle engineering.

2°. A mechanical system with a finite number of degrees of freedom models the vehicle. The road is represented as a stationary random process, continuous, with finite mean square. The vehicle is supposed to move with a constant speed v . Only the linear model is used. In this case the equation of motion is [4]:

$$(1) \quad \mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t),$$

where $\mathbf{M}$ is the $(m \times m)$ matrix of masses and inertia moments, $\mathbf{D}$ is the $(m \times m)$ damping matrix, $\mathbf{K}$ is the $(m \times m)$ rigidity matrix, $\mathbf{p}(t)$ is the m -vector of perturbations and $\mathbf{x}$ is the m -vector of the generalized coordinates.

The equation (1) can be also written in the following equivalent form :

$$(2) \quad \ddot{x}_i + d'_{ij}\dot{x}_j + k'_{ij}x_j = p_i(t), \quad i = 1, \dots, m.$$

Let's consider the functions h_{ij} , $i = 1, \dots, m$, $j = 1, \dots, m$, which verify

$$(3) \quad \ddot{h}_{ij} + d'_{il}\dot{h}_{lj} + k'_{il}h_{lj} = \delta_{ij} \delta(t),$$

δ_{ij} the symbol of Kronecker, $\delta(t)$ the Dirac's distribution, as well as the conditions

$$(4) \quad \begin{cases} \lim_{t \rightarrow \infty} h_{ij}(t) = 0, \\ h_{ij}(t) = 0 \text{ for } t < 0. \end{cases}$$

The matrix $(h_{ij}(t))_{i=1, \dots, m, j=1, \dots, m}$ is called the weighting matrix of the system (2) and it is a fundamental particular solution of (2). Then a particular solution of (2) can be written [3]:

$$(5) \quad x_i(t) = h_{ij} * p_j = \int_{-\infty}^{\infty} h_{ij}(\tau) p_j(t - \tau) d\tau \quad i = 1, \dots, m.$$

The matrix $(H_{ij}(\omega))_{i=1, \dots, m, j=1, \dots, m}$ of the Fourier transforms of $(h_{ij}(t))_{i=1, \dots, m, j=1, \dots, m}$ is called the transfer matrix of the system (2).

Calculating the means, the correlation and the intercorrelation functions of the solutions x_i , using (5) and considering $p_i(t)$ stationary and stationary correlated, it is easy to see that x_i are also stationary and stationary correlated. Then the spectral densities of the solutions can be calculated:

$$\begin{aligned} (6) \quad S_{x_i x_j}(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{p_l p_k}(\tau + \tau_1 - \tau_2) h_{il}(\tau_1) h_{jk}(\tau_2) d\tau_1 d\tau_2 \right) e^{-i\omega\tau} d\tau = \\ &= \int_{-\infty}^{\infty} h_{il}(\tau_1) e^{i\omega\tau_1} d\tau_1 \int_{-\infty}^{\infty} h_{jk}(\tau_2) e^{-i\omega\tau_2} d\tau_2 \left(\frac{1}{\pi} \int_{-\infty}^{\infty} k_{p_l p_k}(\tau + \tau_1 - \tau_2) e^{-i(\tau + \tau_1 - \tau_2)\omega} d\tau \right) = \\ &= H_{il}(-\omega) H_{jk}(\omega) S_{p_l p_k}(\omega) = \bar{H}_{il}(\omega) H_{jk}(\omega) S_{p_l p_k}(\omega), \end{aligned}$$

where $i = 1, \dots, m$, $j = 1, \dots, m$, $k_{p_l p_k}(\tau)$ is the intercorrelation function of p_l and p_k , and $S_{p_l p_k}(\omega)$ is the spectral density of p_l and p_k .

$S_{x_i x_i}(\omega)$ is called the spectral density of x_i and $S_{x_i x_j}(\omega)$ the cross spectral density of x_i and x_j .

If we consider the matrices $S_X(\omega) = (S_{x_j x_i}(\omega))_{i=1, \dots, m, j=1, \dots, m}$ and $S_P(\omega) = (S_{p_j p_i}(\omega))_{i=1, \dots, m, j=1, \dots, m}$ the relation (6) becomes:

$$(7) \quad S_X(\omega) = H(\omega) S_P(\omega) H^*(\omega).$$

By the help of the matrix $S_X(\omega)$, the mean square accelerations can be calculated [1], and this is important in the road vehicle dynamics.

$$(8) \quad \sigma_{\ddot{x}_i} = \int_0^{\infty} \omega^4 S_{x_i x_i}(\omega) d\omega.$$

3°. An elementary model which emphasizes the coupling effect is the model with two degrees of freedom (fig. 1) obtained by simplifying more complex models, as for instance the model with seven degrees of freedom. In this case, the equations of motion are :

$$(9) \quad \begin{cases} \ddot{x}_1 + \frac{1-\gamma}{\frac{1}{\varepsilon} + \frac{\gamma}{\varepsilon}} \ddot{x}_2 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 = 2\zeta_1\omega_1\dot{x}_{01} + \omega_1^2 x_{01}, \\ \ddot{x}_2 + \frac{1-\gamma}{\varepsilon + \gamma} \ddot{x}_1 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 = 2\zeta_2\omega_2\dot{x}_{02} + \omega_2^2 x_{02} \end{cases}$$

where parameters γ , ε , ω_1 , ω_2 , ζ_1 , ζ_2 are : $\gamma = \frac{\rho^2}{ab}$ the coupling coefficient ; $\varepsilon = \frac{a}{b}$ the asymmetry coefficient ; $\omega_i^2 = \frac{k_i}{m_i}$ the partial frequencies ; $2\zeta_i\omega_i = \frac{c_i}{m_i}$, ζ_i the relative dampings ; ρ the inertia radius ; $m_1 = \frac{m(b^2 + \rho^2)}{L^2}$ and $m_2 = \frac{m(a^2 + \rho^2)}{L^2}$ the equivalent masses of the front-side of the vehicle, respectively of the rear-side of the vehicle and a , b , L , x_1 , x_2 , x_{01} , x_{02} , k_1 , k_2 , c_1 , c_2 have the signification given in fig. 1.

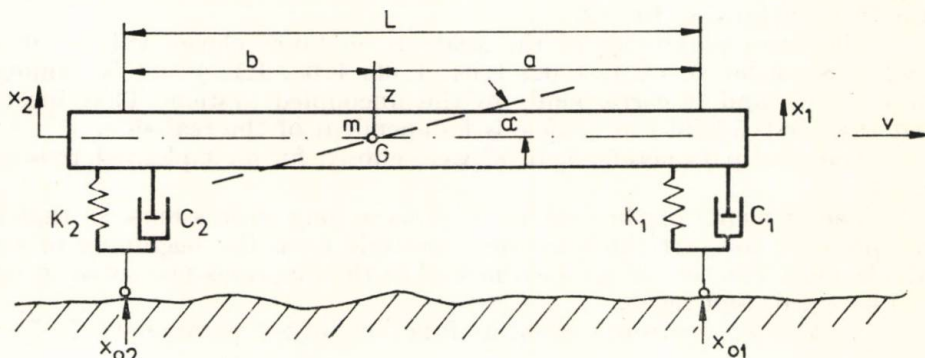


Fig. 1.

Only the stabilized part of the motion will be studied so that it will be taken into account only the asymptotically stable in the large in mean square solution of the system (9).

By means of experiences it was found out that a certain category of roads may be represented by random processes with correlations of this type : $k_{x_0}(\tau) = \sigma_{x_0}^2 e^{-\alpha|\tau|} \cos \beta\tau$, where $\alpha = av$, $\beta = bv$, $a, b, \sigma_{x_0}^2$ being parameters

which can be established by processing the profile of the road. The spectral density corresponding to this correlation function after limiting the spectral belt is :

$$(10) \quad S_{x_0}(\omega) = \begin{cases} \frac{\sigma_{x_0}^2(\omega^2 + \alpha^2 + \beta^2)}{(\omega^2 - \alpha^2 - \beta^2) + 4\alpha^2\omega^2} & \text{for } |\omega| \leq \Omega, \\ 0 & \text{for } |\omega| > \Omega. \end{cases}$$

Considering x_{01} a random process of this type and the relation between x_{01} and x_{02} , $x_{02}(t) = x_{01}(t - T)$, $T = \frac{L}{V}$, the matrix $S_X(\omega)$ and σ_{x_1} , σ_{x_2} can be calculated with (8).

To see the influence of the coupling coefficient on the level of vibrations a numerical analysis was made, considering γ , ε , ω_1 , ω_2 , τ_1 , τ_2 parameters and L and v constants.

All the values that were given were possible or real, taken from the road vehicle engineering.

The values for the constants L and v are $L = 80$ km/h and $L = 2.441$ m.

The behaviour of the mechanical system in ζ_1 and ζ_2 is well known so that only two values, 0.15 and 0.20, were considered. These values are very close to the optimal value.

Three values were considered for ε , one of them being the value of longitudinal symmetry. There values are : 0.95, 1, 1.05. ω_1 and ω_2 take also three values 8, 10, 12.

The main parameter of the analysis, γ , takes eleven values : 0.9, 0.92, 0.84, 0.96, 0.98, 1, 1.02, 1.04, 1.06, 1.08, 1.1. $\gamma = 1$ is among these values and it corresponds to the uncoupled system. That means that the front-side of the vehicle is independent of the rear-side.

The road parameters, a , b , σ^2 were chosen for an asphalted road as in [2].

The elements of the matrix $H(\omega)$ have long expressions, so that it was preferred to start the numerical analysis from the beginning of the calculations. The results are synthesized in the diagrams presented in fig. 2 a) b), c), d) and fig. 3.

σ_{x_1} and σ_{x_2} are represented as functions of γ , parameters ζ_1 , ζ_2 , ε being constants (fig. 2). In fig. 3 it is represented the spectral density for $\omega_1 = \omega_2 = 10$, $\varepsilon = 0.95$, $\zeta_1 = \zeta_2 = 0.15$, and for two distinct values of γ (0.9, 1). It can be noticed (fig. 2) that the diagrams are gathered round three values, in each case, these values corresponding to the three partial frequencies, for $\gamma = 1$. σ_{x_1} is an increasing function of γ , σ_{x_2} is a decreasing function of γ .

4°. For equal and greater frequencies (10, 12) the maximum of σ_{x_2} is greater than the maximum of σ_{x_1} . This is a surprising result, especially

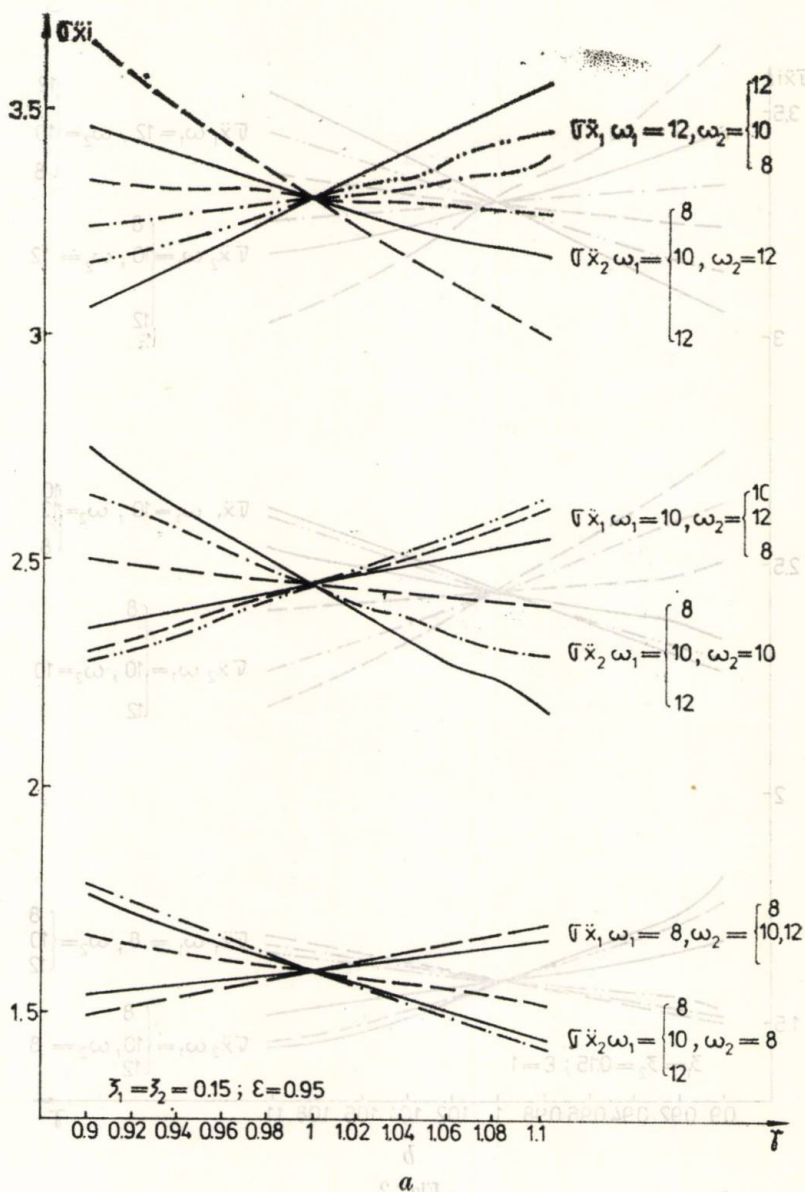


Fig. 2

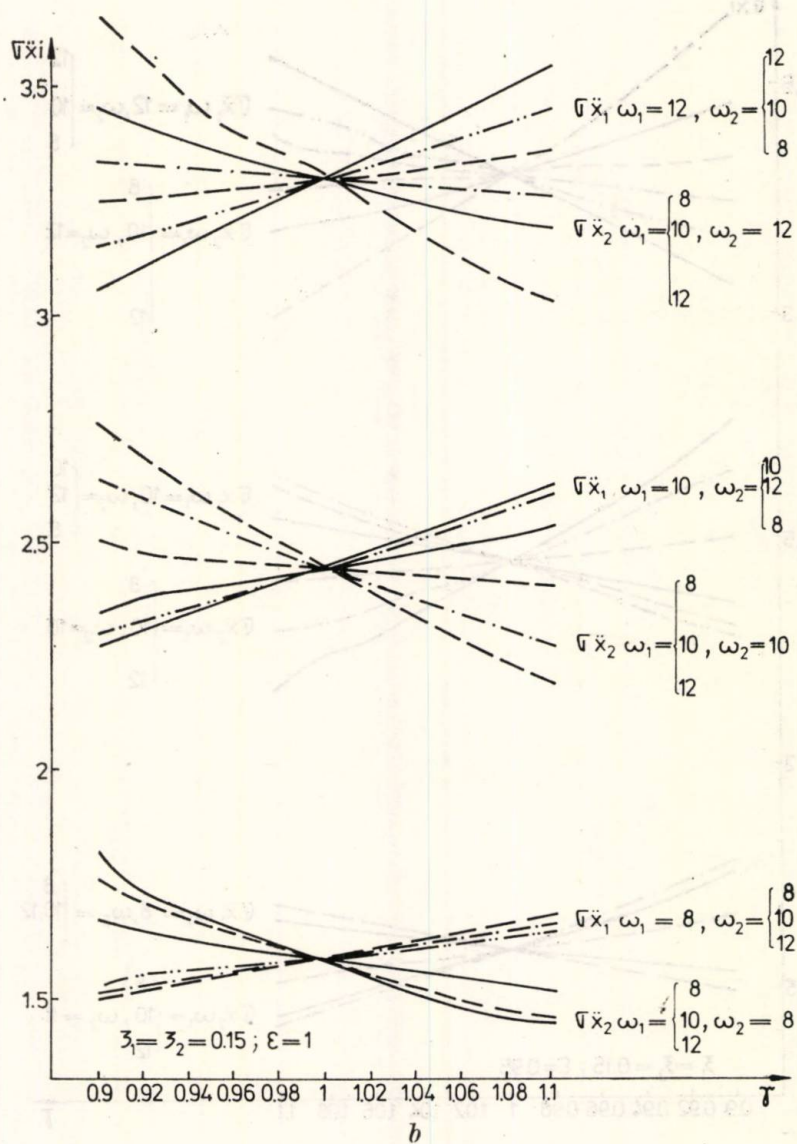


Fig. 2

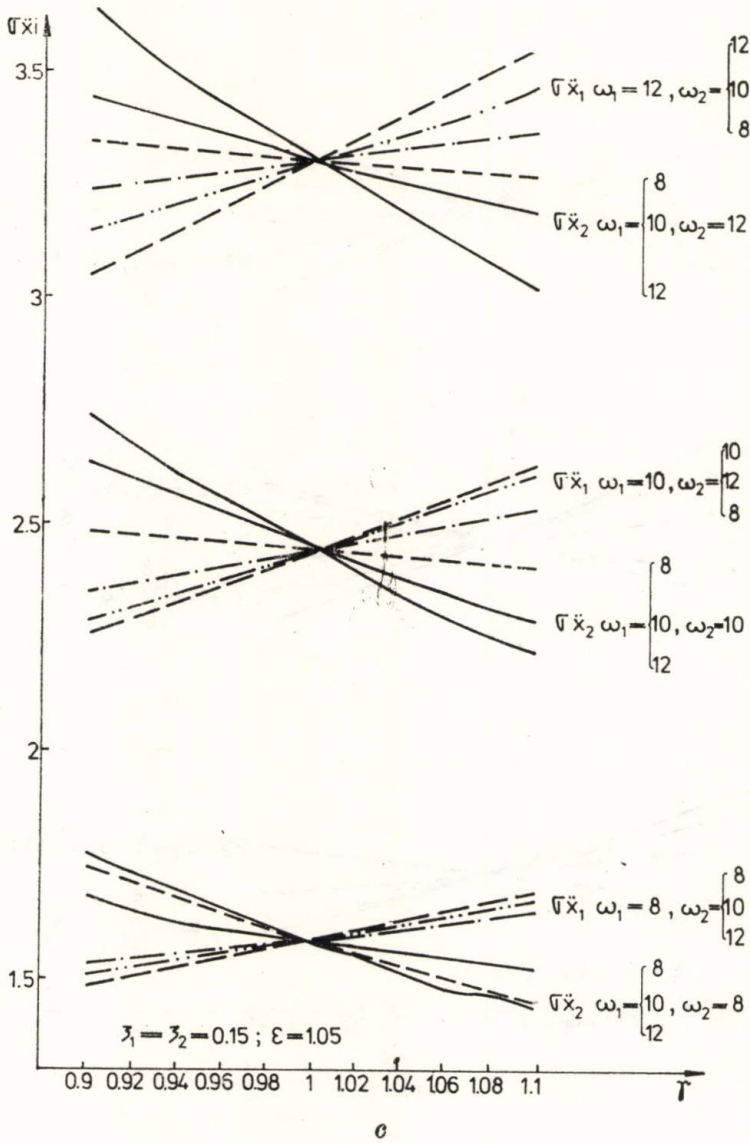


Fig. 2.

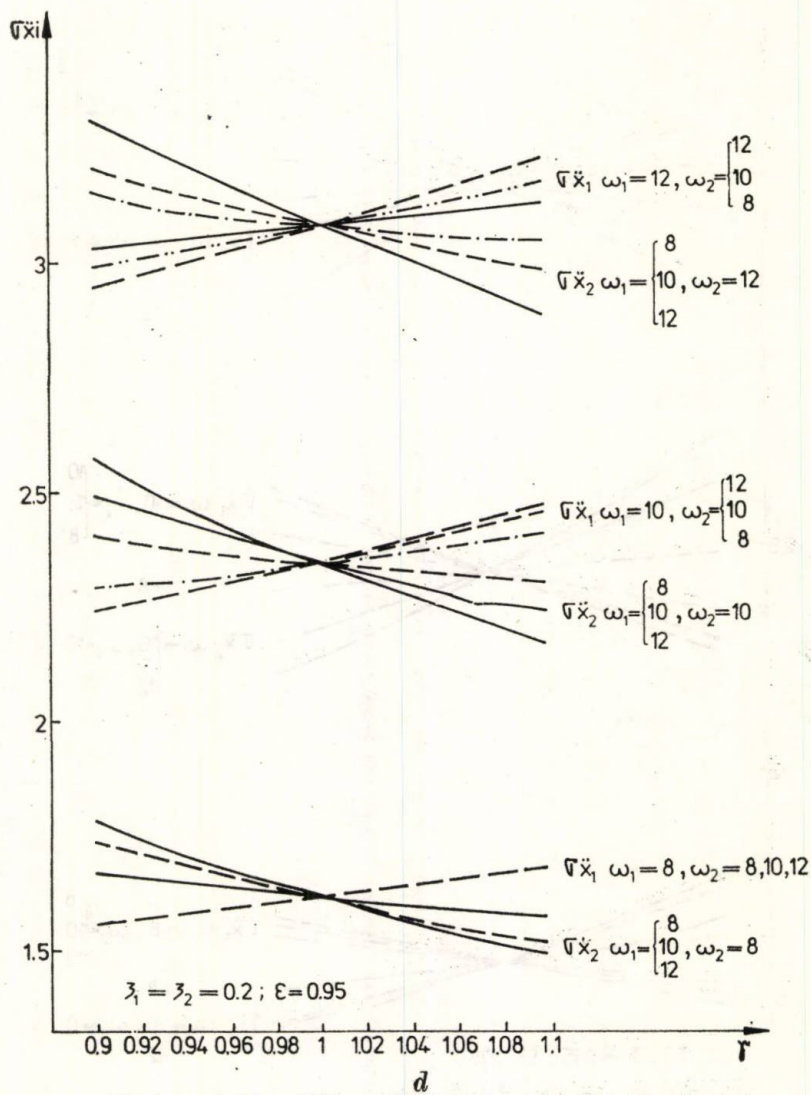


Fig. 2.

for $\varepsilon = 1$, but it can be explained. The rear-side of the vehicle receives its own perturbation from the road, but in the same time through the coupling with the front side it receives a part of its perturbation.

For $\gamma > 1$, $\sigma_{x_2} > \sigma_{x_1}$ and for $\gamma < 1$ $\sigma_{x_1} > \sigma_{x_2}$. σ_{x_1} and σ_{x_2} are increasing on ω_1 and ω_2 . The dependence on ε is rather weak (fig. 2a), b), c).

We infer some practical conclusions from these observations.

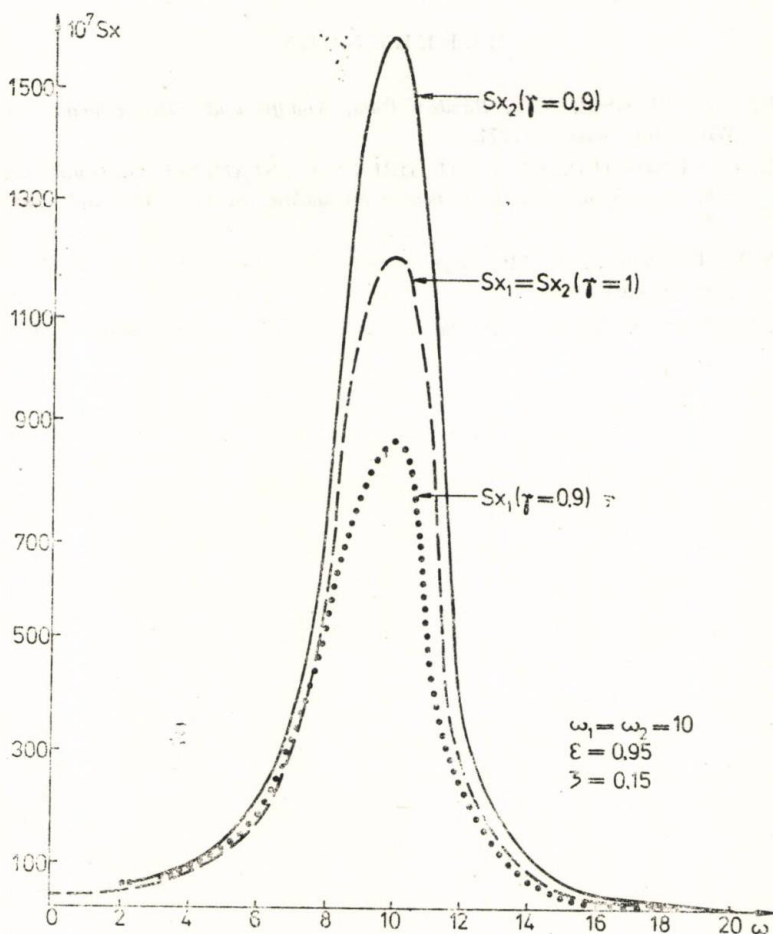


Fig. 3.

If it is necessary for a part of the vehicle to offer a greater comfort indifferent of what happens elsewhere (for example a sanitary car) the parameter γ may be handled in the suitable way. But if a uniform comfort is needed, then it is better to achieve $\gamma = 1$.

The same conclusions can be drawn from the analysis of fig. 3, where a uniform level of vibration is obtained for $\gamma = 1$.

The figures 2 a) and 2 d) show how strong the effect of relative dampings ζ_1 and ζ_2 is : when $\zeta_1 = \zeta_2$ decreases from 0.2 to 0.15, for $\omega = 10$ or $\omega = 12$ a higher level of vibrations can be noticed.

Beside the relative damping, the coupling coefficient is an efficient way to check the level of vibrations.

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*Faculty of Mathematics
University of Bucharest*

1. Introduction

Les systèmes de transmission des données, les systèmes de communication ordinateur-ordinateur, les communications spatiales utilisent, pour réaliser une transmission sûre, des techniques spéciales afin de combattre l'influence des erreurs de communication (error control techniques).

On utilise trois catégories de telles techniques :

- la classe des schémas de correction des erreurs (forward error correction-FEC) — d'ailleurs l'unique méthode possible pour les systèmes qui ne disposent pas d'un canal réaction ;
- la classe des schémas qui réalise la détection des erreurs combinée à la rétransmission automatique sur demande du bloc erroné (automatic repeat request — ARQ) ;
- la classe des schémas hybrides qui utilisent la retransmission combinée à la correction partielle des erreurs, pour la réduction du nombre des demandes de retransmission utilisées (le cas des canaux à taux d'erreur élevé).

Pour la réalisation d'un schéma de contrôle des erreurs, il est nécessaire de connaître aussi bien la nature des erreurs introduits par le canal (les erreurs peuvent apparaître comme des événements indépendants, ou en paquets (burst), ou comme une réunion des deux type (compound channel)), ainsi que les différentes distributions attachées au processus erreur :

- la distribution de la longueur des intervalles entre deux erreurs (gap) ;
- la distribution de la longueur des paquets etc.

La caractérisation statistique des erreurs introduites par le canal peut être réalisée soit par l'enregistrement de la séquence erreur sur un support réel, par exemple bande magnétique, bande perforée etc. et l'interprétation statistique des données enregistrées, assistée par le calculateur, soit par la construction d'un modèle du canal dont on déduit les statistiques qui intéressent.

Pour la caractérisation des erreurs dans le canal numérique, on a proposé différents modèles mathématiques :

- a) les modèles Markov proposés par Gilbert [1], Elliott [2], Fritchman [3], Tsai [4],
- b) les modèles dont les erreurs constituent l'effet d'un processus de renouvellement (proposés par Elliott [2], Sussman, Berger et Mandelbrot),
- c) le modèle Muntner-Wolf — une séquence de N processus de renouvellement [5].

Dans cet article on propose des modèles du canal basés sur un processus de renouvellement Markov qui peut être décrit [11] comme une collection de processus de renouvellement qui se trouvent simultanément en développement, de sorte que les types successifs de renouvellements forment une chaîne Markov.

On décrit d'abord les processus de renouvellement Markov. On présente ensuite une généralisation du modèle Gilbert dans le cas où les états du canal évoluent selon un processus semi-Markov.

On décrit, comme un cas particulier, le canal de renouvellement correspondant au modèle Fritchman à un seul état qui introduit (sûrement) des erreurs.

On propose une partition, de type Fritchman, des états du canal dans la situation du canal à l'évolution semi-markovienne des états.

En conclusion, on analyse le modèle proposé par Muntner et Wolf qui s'avère à être gouverné par un processus semi-Markov dégénéré et on propose une extension aux répartitions plus générales.

2. Processus semi-Markov et processus de renouvellement Markov

On suppose un canal qui effectue des transitions d'un état à l'autre suivant une matrice Markov $P = (p_{ij})$, de dimensions $N \times N$. On considère que le nombre total des états du canal est fini $S = \{1, 2, 3, \dots, N\}$. L'intervalle de temps dont le canal se trouve dans un état i est une variable aléatoire ayant la répartition $F_{ij}(t)$, qui dépend aussi bien de l'état i ainsi que du prochain état visité, j :

$$P\{X_n \leq t \mid J_0, \dots, J_n\} = F_{J_{n-1}, J_n}(t)$$

Par définition, la matrice des probabilités de transition $Q(t)$, de dimension $N \times N$, a l'élément générique :

$$Q_{ij}(t) = p_{ij}F_{ij}(t), \text{ pour } i, j \in S,$$

qui satisfait les relations :

- i) $Q_{ij}(t) = 0$ pour $t < 0$
- ii) $\sum_{j \in S} Q_{ij}(t) = \sum_{j \in S} p_{ij} = 1$ pour $i \in S$.

La matrice Q , dénommée aussi la matrice semi-Markov sur S , induit le processus $\{(J_n, T_n) : n \geq 0\}$, où $T_n = \sum_{i=0}^n X_i$, défini sur un espace probabilisé $(\Omega, \mathcal{F}, P)$ de cette façon :

$$X_0 = 0$$

- 1) $P\{J_0 = i\} = a_i$

(le symbole $I = (a_1, \dots, a_N)$, $a_i \geq 0$, $\sum_{i=1}^N a_i = 1$ représente le vecteur des probabilités initiales),

2) $P\{J_n = i, T_n \leq t | J_0, J_1, X_1, J_2, X_2, \dots, J_{n-1}, X_{n-1}\} = Q_{J_{n-1}, i}(t - T_n)$, pour tout $i \in S$.

On peut démontrer [v. 10] que le processus bidimensionnel $\{J_n, T_n\}$ est un processus Markov et que le processus $\{J_n : n \geq 0\}$ une chaîne Markov avec les probabilités de transition

$$P\{J_n = j | J_0, J_1, \dots, J_{n-1} = i\} = Q_{ij}(\infty) = p_{ij}$$

Le processus $\{J_n, T_n\}$ est dénommé processus de renouvellement Markov. On note $N(t) = \sup \{n \geq 0 : T_n \leq t\}$. On entend par processus semi-Markov déterminé par (N, I, Q) le processus $Z_t = J_{N(t)}$.

Avec J_n et T_n , on a noté l'état visité lors de la n -ème transition et le moment respectif. On dit que le processus semi-Markov (ou le processus de renouvellement Markov) se trouve dans l'état i , si $Z_t = i$.

Cas particuliers

1. On démontre [11] que les moments successifs T_n d'entrée dans un état défini $k - \{T_n : J_n = k\}$ forment une processus de renouvellement (les intervalles de temps entre les moments T_n de l'entrée dans l'état k sont des variables aléatoires indépendantes et identiquement distribuées). Donc il en résulte que si $N = 1$ (le canal a un seul état), le processus de renouvellement Markov devient un processus de renouvellement.

2. Quand $Q_{ij}(t) = p_{ij}U_1(t)$ où

$$U_1(t) \stackrel{\Delta}{=} \begin{cases} 0 & \text{pour } t < 1 \\ 1 & \text{pour } t \geq 1 \end{cases}$$

le processus de renouvellement Markov devient une chaîne Markov discrète.

3. Le processus Markov en temps continu est un processus de renouvellement Markov pour lequel toutes les distributions $Q_{ij}(t)$ sont de la forme :

$$Q_{ij}(t) = p_{ij}(1 - e^{-q_i t}), \quad p_{ii} = 0, \quad q_i > 0, \quad i \in S,$$

c'est-à-dire le temps de maintien dans l'état i dépend seulement de l'index i et a une distribution exponentielle.

Pour décrire un processus semi-Markov à temps discret, nous définirons les fonctions de répartition de probabilité suivantes [9] :

$$i) h_{ij}(m) = P\{X_n = m | J_{n-1} = i, J_n = j\}, \quad m = 1, 2, 3, \dots, \quad i, j \in S$$

c'est-à-dire la probabilité du maintien dans l'état i pour m unités de temps si la transition suivante sera vers l'état j .

Supposons $h_{ij}(0) = 0$, $\sum_{m=0}^{\infty} m h_{ij}(m) < \infty$, $\forall i, j \in S$;

ii) $W_i(m) = P\{X_n = m | J_0, \dots, J_{n-1} = i\}$,

c'est-à-dire la probabilité du maintien du canal dans l'état i pour un temps égal à m unités, l'état suivant étant un état quelconque (de S).

Entre $W_i(m)$ et $h_{ij}(m)$ s'établit la relation :

$$W_i(m) = \sum_{j=1}^N p_{ij} h_{ij}(m).$$

Un processus semi-Markov discret peut être défini aussi de la façon suivante. On spécifie $W_i(m)$ pour tous les états. Pour la description complète du processus, on définit encore les probabilités des transitions dans chaque état prochain, conditionnées par le temps passé dans l'état présent :

$$p_{ij}(m) = P\{J_n = j | X_n = m, J_{n-1} = i\} \quad i, j \in S \quad m = 1, 2, 3, \dots$$

Dans l'article présent nous utiliserons la probabilité de l'évènement : le canal qui est entré au moment $t = 0$ dans l'état i effectue la transition suivante dans l'état j au moment $\dots$. Nous désignons cet événement par $c_{ij}(m)$:

$$c_{ij}(m) = P\{J_n = j | X_n = m, J_{n-1} = i\}$$

On peut démontrer immédiatement les relations :

$$c_{ij}(m) = p_{ij} h_{ij}(m) = W_i(m) p_{ij}(m) \quad (1)$$

Un processus semi-Markov indépendant [9], [12] est un processus semi-Markov pour lequel $p_{ij}(m) = p_{ij}$. D'après (1) résulte $h_{ij}(m) = W_i(m)$. Un exemple de processus semi-Markov indépendant est la chaîne Markov pour laquelle

$$h_{ij}(m) = W_i(m) = \delta(m - 1) = \begin{cases} 1 & \text{pour } m = 1 \\ 0 & \text{pour } m \neq 1 \end{cases}$$

Si le processus semi-Markov entre dans l'état i au moment zéro et si l'on désigne par Z l'état du processus au moment t , on démontre que pour les processus semi-Markov ergodiques la probabilité que le processus se trouve dans l'état j au moment t , lorsque t tend vers l'infini, est indépendante de i et on la désigne par l'expression :

$$\lim_{t \rightarrow \infty} P\{Z_t = j | Z_0 = i\} = \frac{\pi_j \bar{\tau}_j}{\sum_{k=1}^N \pi_k \bar{\tau}_k} = \frac{\pi_j \bar{\tau}_j}{\bar{\tau}} \stackrel{\Delta}{=} \theta_j,$$

où $\bar{\tau}_j = E(X_n | J_{n-1} = j)$.

$\pi = (\pi_1, \dots, \pi_j, \dots, \pi_N)$ est le vecteur qui indique la répartition stationnaire. Le vecteur π est désigné par la relation

$$\pi P = \pi, \quad \sum_{j=1}^N \pi_j = 1.$$

Lorsqu'on commence l'observation d'un processus semi-Markov à partir d'un moment quelconque, qui ne coïncide pas strictement à une transition, il faut considérer les répartitions ${}_r p_{ij}$ et ${}_r h_{ij}(m)$, à la place des répartitions p_{ij} et $h_{ij}(m)$, car l'origine est un instant arbitraire. Par ${}_r p_{ij}$ on a désigné la probabilité que la première transition ait lieu de l'état i , où se trouvait le processus au moment du commencement de l'observation, dans l'état j . ${}_r h_{ij}(m)$ indique la probabilité que le temps de maintien resté jusqu'à cette transition soit m unités. Ces quantités sont données par les expressions [9]:

$${}_r p_{ij} = p_{ij} \frac{\bar{\tau}_{ij}}{\bar{\tau}_i},$$

$${}_r h_{ij}(m) = \frac{\sum_{k=m}^{\infty} h_{ij}(k)}{\bar{\tau}_{ij}} \Delta = \frac{{}_* h_{ij}(m-1)}{\bar{\tau}_{ij}},$$

$$\text{où } \bar{\tau} = \sum_{m=0}^{\infty} h_{ij}(m)m.$$

On obtient :

$${}_r c_{ij}(m) = \frac{p_{ij} {}_* h_{ij}(m-1)}{\bar{\tau}_i}.$$

3. Canal stationnaire additif de type Gilbert à l'évolution semi-markovienne des états

Un tel canal satisfait aux conditions suivantes :

a) Il existe un vecteur qui donne les probabilités d'occupation par le canal des états de l'ensemble S à tout moment, vecteur indépendant de l'état initial :

$$\tilde{p}_0 = (\theta_1, \dots, \theta_N), \quad \sum_{j=1}^N \theta_j = 1, \quad \theta_j > 0$$

$$\theta_j = \frac{\pi_j \bar{\tau}_j}{\sum_{j=1}^N \pi_j \bar{\tau}_j} = \frac{\pi_j \bar{\tau}_j}{\bar{\tau}}$$

La probabilité de la succession des états du canal

$$\overbrace{Z_1 \dots Z_1}^{m_1}, \overbrace{Z_2 \dots Z_2}^{m_2}, \dots, \overbrace{Z_n \dots Z_n}^{m_n} \triangleq Z_1^{m_1} Z_2^{m_2} \dots Z_n^{m_n}$$

est donnée par la relation :

$$\begin{aligned} p([Z_1^{m_1} Z_2^{m_2} \dots Z_n^{m_n}]) &= \tilde{p}(Z_1^{m_1} Z_2^{m_2} \dots Z_n^{m_n}) = \\ &= \theta_{Z_1, c_{Z_1 Z_2}}(m_1) \dots c_{Z_{n-1} Z_n}(m_{n-1}) * W_{Z_n}^{(m_n)}. \end{aligned}$$

b) Il existe une matrice stochastique $p(e/Z)$, $e \in A$, $Z \in S$, ainsi que la probabilité de n'importe quelle succession d'erreur $e_1 \dots e_n$ soit conditionnée par la succession des états du canal conformément à l'égalité

$$p(e_1 \dots e_n / Z_1, \dots, Z_n) = p(e_1 / Z_1) \dots P(e_n / Z_n)$$

pour tous les $e_i \in A$, $Z_i \in S$, $i = 1, 2, \dots, n$

c) La probabilité p sur $\mathcal{F}_A$ est entièrement définie par la probabilité $p(e/Z)$ conformément à l'égalité :

$$\begin{aligned} p([e_1 \dots e_n]) &= \sum_{Z_1, \dots, Z_n \in S} p(e_1 \dots e_n / Z_1 \dots Z_n) \tilde{p}(Z_1, \dots, Z_n) = \\ &= \sum_{Z_1, \dots, Z_n \in S} p(e_1 / Z_1) \dots p(e_n / Z_n) \tilde{p}(Z_1 \dots Z_n). \end{aligned}$$

Dans le cas particulier d'une chaîne Markov, on obtient avec

$$\theta_j = \pi_j, c_{ij}(m) = p_{ij} \delta(m-1), W_i(m) = \delta(m-1).$$

$$p(Z_1 \dots Z_n) = \pi_{Z_1} (p(Z_2 / Z_1) \dots p(Z_n / Z_{n-1}))$$

qui est l'expression donnée en [7] pour p , dans le cas du canal stationnaire additif du type Gilbert à l'évolution markovienne des états.

Fritchman [3] a élargi les résultats obtenus par Gilbert pour le cas général d'un modèle Markov avec N états d'où un nombre de k états ne produisent pas des erreurs et $(N-k)$ états conduisent certainement à des erreurs. Par conséquent, on a proposé la partition des états du canal en deux sousensembles d'états : états de succès (qui ne conduisent pas à l'apparition des erreurs) et des états d'erreur. Ce modèle a été analysé en [6] pour le cas où la répartition initiale des états est quelconque (pas nécessairement la répartition stationnaire).

Fritchman [3] et Tsai [4] ont accordé une attention particulière au cas qui présente le modèle ayant un seul état d'erreur. Si l'on désigne par $\tau_i^{(l)}$ le moment de la l -ème entrée dans l'état i (que nous considérons comme étant l'unique état qui conduit sûrement à l'erreur, on démontre en [8] que les variables $\tau_i^{(l+1)} - \tau_i^{(l)}$, $l = 1, 2 \dots$ sont indépendantes et identiquement distribuées, et $p(\tau_i^{(l+1)} - \tau_i^{(l)} = n) = f(n, i, i) = \sum_{m \neq i, 1 \leq m \leq n-1} (p(i_1/i) p(i_2/i_1) \dots p(i/i_{n-1}))$. Donc le processus $\{\tau_i^{(l)}, l=1, 2 \dots\}$ est un processus de renouvellement, qui est le même avec le processus erreur. Il s'agit donc d'un modèle Elliott [2] pour des processus erreur du type renouvellement. Calculons dans ce cas la probabilité d'une configuration d'erreurs (error pattern) de la forme $0^a 10^b 10^b \dots 10^{b_{m-1}} 10^c$, qui contient un nombre de m erreurs. Notons :

$$*f(n, i, i) = \sum_{k=n+1} f(k, i, i) = p\{\tau_i^{(l+1)} - \tau_i^{(l)} > n + 1\}$$

la probabilité que l'intervalle entre les erreurs plus 1 (gap) soit plus grand que n et avec

$$\bar{G} = \sum_{n=1}^{\infty} f(n, i, i) n$$

c'est-à-dire la valeur moyenne de la longueur du „gap” (le temps moyen de récurrence), supposée finie.

$1/\bar{G}$ représente la probabilité nonconditionnée qu'un bit soit erroné. La probabilité du bloc $0^a 10^b 10^b \dots 10^{b_{m-1}} 10^c$ est donnée par l'expression :

$$\frac{1}{\bar{G}} *f(a, i, i) \prod_{k=1}^{m-1} f(b_k + 1, i, i) *f(c, i, i)$$

La partition de l'espace des états (de l'ensemble S) en deux sousensembles A et B par l'application de l' S en $\{0, 1\}$ définie par

$$\varphi(i) = \begin{cases} 0 & \text{pour } i \in A \\ 1 & \text{pour } i \in B \end{cases}$$

peut être élargie aussi dans la situation dont l'évolution des états est semi-markovienne.

Si l'on considère $e_n = \varphi(Z_n)$ où avec $\{Z_n : n > 0\}$ on a noté le processus semi-Markov qui décrit l'état du canal pour chaque bit transmis, on obtient le processus erreur $\{e_n, n \geq 0\}$, processus qui peut représenter une classe de canaux plus large que la classe des canaux fonctions de chaîne Markov proposée par Fritchman [3].

4. Une généralisation du modèle Muntner-Wolf

Le modèle suppose un canal de communications additif, symétrique (e_n indépendant de X_n — le symbole d'entrée dans le canal au moment n). Soit T un nombre entier grand mais fini. Les erreurs que l'on trouve dans l'intervalle de $t_0 + lT$ à $t_0 + (l+1)T - 1$ sont générées par le processus (le canal) de renouvellement C_i avec la probabilité λ_i , $i = 1, 2, \dots, N$. Au moment $t_0 + (l+1)T$ un autre processus de renouvellement C_j est choisi avec la probabilité λ_j , $j = 1, 2, \dots, N$ pour générer les erreurs dans ce nouvel intervalle de longueur T . Le choix de C_j est indépendant des processus sélectionnés antérieurement. En plus, les points de sélection du canal, disposés à des distances T sont uniformément répartis : $P\{t_0 = i\}$, $i = 1, 0, \dots, T-1$, est $\frac{1}{T}$. Donc, le canal devient

un autre canal de renouvellement après un nombre de T bits transmis sans être pourtant un canal de renouvellement.

De la description du modèle Muntner-Wolf, que nous avons reproduite selon [5], on observe que l'état du canal (le type du canal) Z à un moment t représente un processus semi-Markov retardé (delayed) et dégénéré décrit par les répartitions suivantes :

$$p_{ij} = \lambda_j \sum_j \lambda_j = 1, \quad i = 1, 2, \dots, N$$

$${}_r p_{ij} = \lambda_j \quad j = 1, 2, \dots, N$$

$$h_{ij}(m) = \delta(m - T) = \begin{cases} 1 & m = T \\ 0 & m \neq T \end{cases}$$

$${}_r h_{ij}(m) = \frac{1}{T}.$$

Naturellement, nous sommes penchés à étendre ce modèle au cas de répartitions générales. Par conséquent, ce modèle plus général offre une meilleure description des canaux réels.

5. Conclusions

Les modèles proposés dans cet article supposent une évolution des états du canal conforme à un processus semi-Markov à temps discret et un nombre fini d'états.

Par la possibilité d'alloquer des répartitions de probabilité arbitraires pour le moment de transition, les modèles proposés sont suffisamment flexibles pour couvrir une classe large de canaux.

L'extension qu'ils permettent ne conduit pas à des difficultés particulières de calcul. La forme récursive des expressions de certains résultats permet la programmation commode sur l'ordinateur.

Les modèles n'essayent pas à représenter des procès physiques qui produisent des erreurs, mais promettent à illustrer la manière dont les erreurs apparaissent en temps.

Faculté d'Electronique et de Télécommunications
L'Institut Politechnique Bucarest

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The first part of the paper is devoted to a general discussion of the problem. It is shown that the problem is of great importance in the theory of the structure of the atom. The second part is devoted to a detailed discussion of the problem. It is shown that the problem is of great importance in the theory of the structure of the atom.

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Le but de cette note est d'exposer une généralisation (v. le théorème) d'un lemme de Kronecker (v. le corollaire 3) et diverses conséquences utiles de cette généralisation.

Partout dans ce qui suit E désignera un espace vectoriel normé quelconque.

Théorème: Soient (a_n) et (b_n) deux suites de scalaires.

Si:

$$1) \lim_{n \rightarrow \infty} |a_n| = \infty$$

2) il existe $M > 0$ tel que $\frac{1}{|a_n|} \sum_{k=1}^{n-1} |b_{k+1} - b_k| \leq M$, pour tout $n \in N$, $n \geq 2$.
alors pour toute série convergente $\sum_{n \geq 1} x_n$ d'éléments de E on a

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n b_k x_k = 0.$$

Démonstration. Notons $S_n = \sum_{k=1}^n x_k$ et par S la somme de la série

$\sum_{n \geq 1} x_n$. Alors

$$\begin{aligned} (*) \quad \sum_{n=1}^n b_n x_n &= b_1 S_1 + \sum_{k=2}^n b_k (S_k - S_{k-1}) = b_n S_n - \sum_{k=1}^{n-1} (b_{k+1} - b_k) S_k = \\ &+ b_n (S_n - S) + b_1 S - \sum_{k=1}^{n-1} (b_{k+1} - b_k) (S_k - S). \end{aligned}$$

Soit $\varepsilon > 0$. Puisque $\lim_{n \rightarrow \infty} (S_n - S) = 0$, il existe $m \in N^*$ tel que $\|S_k - S\| < \frac{\varepsilon}{2M}$, pour tout $k \geq m$. Alors pour tout $n > m + 1$ on a

$$\begin{aligned} \left\| \frac{1}{a_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) (S_k - S) \right\| &\leq \left\| \frac{1}{a_n} \sum_{k=1}^{m-1} (b_{k+1} - b_k) (S_k - S) \right\| + \\ &+ \frac{1}{|a_n|} \sum_{k=m}^{n-1} |b_{k+1} - b_k| \|S_k - S\| < \left\| \frac{1}{a_n} \sum_{k=1}^{m-1} (b_{k+1} - b_k) (S_k - S) \right\| + \\ &+ M \cdot \frac{\varepsilon}{2M} \end{aligned}$$

Mais $\lim_{n \rightarrow \infty} \frac{1}{a_n} = 0$, donc il existe $n_1 > m + 1$ tel que pour tout $n \geq n_1$ on ait

$$\frac{1}{|a_n|} \left\| \sum_{k=1}^{n-1} (b_{k+1} - b_k) (S_k - S) \right\| < \frac{\varepsilon}{2}, \text{ donc } \left\| \frac{1}{a_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) (S_k - S) \right\| < \varepsilon$$

ce qui signifie que

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) (S_k - S) = 0.$$

Puis, pour tout $n > 1$ on a

$$\frac{|b_n|}{|a_n|} = \frac{1}{|a_n|} \left| b_1 + \sum_{k=1}^{n-1} (b_{k+1} - b_k) \right| \leq \frac{|b_1|}{|a_n|} + \frac{1}{|a_n|} \cdot \sum_{k=1}^{n-1} |b_{k+1} - b_k| \leq \frac{|b_1|}{|a_n|} + M.$$

Comme $\lim_{n \rightarrow \infty} \frac{|b_1|}{|a_n|} = 0$, il en résulte que la suite $\left(\frac{b_1}{a_n} \right)_n$ est bornée et donc la suite $\left(\frac{b_n}{a_n} \right)_n$ est aussi bornée. Ainsi

$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} (S_n - S) = 0$ et par suite, en vertu de la relation (*), on a

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n b_k x_k = 0.$$

Corollaire 1. Soient (a_n) une suite de nombres réels et (b_n) une suite de scalaires. Si :

- 1) la suite $\left(\frac{b_{n+1} - b_n}{a_{n+1} - a_n} \right)_n$ est bornée,
- 2) la suite (a_n) est strictement monotone et $\lim_{n \rightarrow \infty} |a_n| = \infty$, alors pour toute série convergente $\sum_{n \geq 1} x_n$ d'éléments de E on a

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n b_k x_k = 0$$

Démonstration. Comme la suite $\left(\frac{a_1}{a_n} \right)_n$ converge vers zéro, elle est

bornée et comme, par l'hypothèse, la suite $\left(\frac{b_{n+1} - b_n}{a_{n+1} - a_n}\right)_n$ est aussi bornée, il en résulte qu'il existe $M > 0$ tel que

$$\left|\frac{b_{n+1} - b_n}{a_{n+1} - a_n}\right| \leq M \text{ et } \left|\frac{a_1}{a_n}\right| \leq M, (\forall) n \in N^*.$$

Alors pour tout $n \geq 2$ on a

$$\frac{1}{|a_n|} \sum_{k=1}^{n-1} |b_{k+1} - b_k| = \sum_{k=1}^{n-1} \left| \frac{b_{k+1} - b_k}{a_{k+1} - a_k} \right| \cdot \frac{|a_{k+1} - a_k|}{|a_n|} \leq \frac{M}{|a_n|}.$$

$$\sum_{k=1}^{n-1} |a_{k+1} - a_k| \leq M \frac{|a_n| + |a_1|}{|a_n|} \leq M(1 + M)$$

et le théorème entraîne le corollaire.

Remarque. Le théorème et le corollaire 1 restent vrais, avec la même démonstration, si on suppose que E est une algèbre normée et que (b_n) est une suite d'éléments de E .

Corollaire 2. Soient (a_n) une suite de scalaire et (b_n) une suite monotone de nombres réels. Si :

1) $\lim_{n \rightarrow \infty} |a_n| = \infty,$

2) la suite $\left(\frac{b_n}{a_n}\right)_n$ est bornée,

alors pour toute série convergente $\sum_{n \geq 1} x_n$ d'éléments de E on a

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n b_k x_k = 0$$

Démonstration. Compte tenu du fait que $\lim_{n \rightarrow \infty} \frac{b_1}{a_n} = 0$, on en déduit que la suite $\left(\frac{b_1}{a_n}\right)_n$ est bornée et comme la suite $\left(\frac{b_n}{a_n}\right)_n$ est aussi bornée, on en conclut qu'il existe $M > 0$ tel que

$$\frac{1}{|a_n|} \sum_{k=1}^{n-1} |b_{k+1} - b_k| = \frac{|b_n - b_1|}{|a_n|} \leq \frac{|b_n|}{|a_n|} + \frac{|b_1|}{|a_n|} \leq M, (\forall) n \geq 2, n \in N^*,$$

donc toutes les hypothèses du théorème sont vérifiées.

Corollaire 3: Si (a_n) est une suite monotone de nombres réels, telle que $\lim_{n \rightarrow \infty} |a_n| = \infty$, alors pour toute série convergente $\sum_{n \geq 1} x_n$ d'éléments de E on a

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n a_k x_k = 0$$

Démonstration. C'est une conséquence immédiate du corollaire 2.

Remarque. Dans le cas $E = \mathbb{R}$ le corollaire 3 représente même le lemme de Kronecker.

Corollaire 4. Soient (a_n) une suite strictement croissante de nombres réels et (b_n) une suite d'éléments de E . Si:

$$1) \lim_{n \rightarrow \infty} a_n = \infty, \quad 2) \lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l,$$

$$\text{alors } \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = l.$$

Démonstration. Notons $x_n = \frac{b_{n+1} - b_n}{a_{n+1} - a_n}$ et $y_n = \frac{b_n}{a_n}$, pour tout $n \in \mathbb{N}^*$. Alors on a

$$\begin{aligned} x_n - y_n &= x_n - \frac{1}{a_n} \cdot \sum_{k=1}^{n-1} (b_{k+1} - b_k) - \frac{b_1}{a_n} = x_n - \frac{1}{a_n} \cdot \sum_{k=1}^{n-1} (a_{k+1} - a_k) x_k - \\ &\quad - \frac{b_1}{a_n} = -\frac{b_1}{a_n} + \frac{1}{a_n} \left[a_1 x_1 + \sum_{k=1}^{n-1} a_{k+1} (x_{k+1} - x_k) \right], \end{aligned}$$

donc, en vertu du corollaire, 3, il en résulte que $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$, car la série

$$x_1 + \sum_{k=1}^{\infty} (x_{k+1} - x_k)$$

est convergente. Par conséquent, $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} x_n = l$.

Remarque. Dans le cas $E = \mathbb{R}$, le corollaire 4 s'appelle le lemme de Stolz.

Corollaire 5: Si (x_n) est une suite d'éléments de E , convergente vers x , alors la suite (y_n) , où $y_n = \frac{1}{n} \cdot \sum_{k=1}^n y_k$, est aussi convergente vers x .

Démonstration. En effet, pour toute $n \in N^*$ on a

$$x_{n+1} - y_n = \frac{1}{n} (nx_{n+1} - x_n - x_{n-1} - \dots - x_1) = \frac{1}{n} \cdot \sum_{k=1}^n k(x_{k+1} - x_k),$$

et comme la série $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ est convergente, on en déduit, en tenant compte du corollaire 3, que $\lim_{n \rightarrow \infty} (x_{n+1} - y_n) = 0$, donc $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} x_n = x$.

Évidemment, ce corollaire se peut aussi démontrer en utilisant le corollaire 4 si on pose $b_n = \sum_{k=1}^n x_k$ et $a_n = n$, pour tout $n \in N^*$.

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*Faculté de Mathématiques
Université de Bucarest*

GAUSS-MARKOV ESTIMATION FOR MULTIVARIATE LINEAR MODELS: A CLASSICAL APPROACH

LUIS ANTONIO PEREZ GONZALES

Introduction and summary

In this paper we present a multivariate extension of the classical GAUSS-MARKOV Theorem [9]. For this, a Multivariate — Least Square Method for random matrices is introduced and an estimable matrix function is defined. Some considerations on these estimable matrix functions are also presented. The last considerations are based on direct generalized extensions to some known distribution functions, namely Normal, Wishart and F distributions. A brief introduction to these distributions is given.

§ 1. Preliminary notions and notations

Let $\mathbf{B} = (b_{ik})_{1 \leq i \leq q, 1 \leq k \leq N}$ be a $q \times N$ -matrix and $\mathbf{E}' = (\mathbf{b}_1, \dots, \mathbf{b}_q)$, $\mathbf{b}'_i = (b_{i1}, \dots, b_{iN})$, $1 \leq i \leq q$, i.e. $\mathbf{b}'_i$ denotes the i -th row of $\mathbf{B}$.

Definition 1.1. Define the matrix of matrices $\mathbf{W}(\mathbf{B})$ as

$$\mathbf{W}(\mathbf{B}) = \mathbf{B} \Delta \mathbf{B}' = (\mathbf{b}_i \mathbf{b}'_j)_{1 \leq i, j \leq q} \quad (1.1)$$

where $\mathbf{W}(\mathbf{B})$ is a square matrix $qN \times qN$

Definition 1.2. Let $\mathbf{C} = (C_{mj})_{1 \leq m \leq r, 1 \leq j \leq q}$ be a $r \times q$ -matrix, $\mathbf{G}$ a $N \times s$ -matrix and $\mathbf{W}(\mathbf{B})$ the matrix of matrices defined by (1.1).

We shall define the products $\mathbf{C} \cdot \mathbf{W}(\mathbf{B}) \cdot \mathbf{C}'$ and $\mathbf{G}' \cdot \mathbf{W}(\mathbf{B}) \cdot \mathbf{G}$ as

$$\mathbf{C} \cdot \mathbf{W}(\mathbf{B}) \cdot \mathbf{C}' = \left(\sum_{j=1}^q \sum_{m=1}^r C_{kj} C_{im} b_j b'_m \right)_{1 \leq i, k \leq r}$$

$$\mathbf{G}' \cdot \mathbf{W}(\mathbf{B}) \cdot \mathbf{G} = (\mathbf{G}'(\mathbf{b}_j \mathbf{b}'_m))_{1 \leq j, m \leq q}$$

It is easy to verify the following properties :

P.1. $\mathbf{W}(\mathbf{CBG}) = \mathbf{G}'\mathbf{B}' \cdot \mathbf{W}(\mathbf{C}) \cdot \mathbf{BG} = \mathbf{CB} \cdot \mathbf{W}(\mathbf{G}) \cdot \mathbf{B}'\mathbf{C}' = \mathbf{C} \cdot (\mathbf{G}' \cdot \mathbf{W}(\mathbf{B}) \cdot \mathbf{G}) \cdot \mathbf{C}'$

P.2. If $\mathbf{W}(\mathbf{B})$ can be written in the form $\mathbf{A} \otimes \mathbf{B}$ where $\mathbf{A}$ is $q \times q$, $\mathbf{B}$ is $N \times N$ and $\mathbf{A} \otimes \mathbf{D} = (a_{ij} \mathbf{D})_{1 \leq i, j \leq q}$ is the Kronecker product, then

$$\mathbf{W}(\mathbf{CB}) = (\mathbf{CAC}') \otimes \mathbf{D}$$

$$\begin{aligned} \mathbf{W}(\mathbf{B}\mathbf{G}) &= \mathbf{A} \otimes (\mathbf{G}'\mathbf{D}\mathbf{G}) \\ \mathbf{W}(\mathbf{C}\mathbf{B}\mathbf{G}) &= (\mathbf{C}\mathbf{A}\mathbf{C}') \otimes (\mathbf{G}'\mathbf{D}\mathbf{G}) \end{aligned} \quad (1.2)$$

(from now, products of the form $(\mathbf{C}\mathbf{A}\mathbf{C}') \otimes (\mathbf{G}'\mathbf{D}\mathbf{G})$ will be denoted by $\mathbf{C}\mathbf{A}\mathbf{C}' \otimes \mathbf{G}'\mathbf{D}\mathbf{G}$)

Assume that $\mathbf{Y}$ is a $q \times N$ random matrix $\mathbf{Y} = (\mathbf{Y}_{ij})_{1 \leq i \leq q, 1 \leq j \leq N}$.

Denote by $E(\mathbf{Y})$ the expectation of $\mathbf{Y}$, as the matrix having the elements $E(\mathbf{Y}_{ij})$.

Definition 1.3. The variance of the random matrix $\mathbf{Y}$ is the $qN \times qN$ matrix defined as

$$\mathbf{D}^2(\mathbf{Y}) = E[\mathbf{W}(\mathbf{Y} - E(\mathbf{Y}))]$$

Definition 1.4. A random matrix $\mathbf{Y}$, $q \times N$, taking values $\mathbf{y} = (y_{ij})_{1 \leq i \leq q, 1 \leq j \leq N}$ in $R^{q \times N}$ is said to have a generalized normal distribution $N(\mathbf{U}, \mathbf{D})$ if its probability density function can be written as

$$h_Y(\mathbf{y}) = \frac{1}{(2\pi)^{qN/2} |\mathbf{D}|^{1/2}} e^{-\frac{1}{2} \text{tr} \mathbf{D}^{-1} \mathbf{W}(\mathbf{y})}$$

where $\mathbf{U} \in R^{q \times N}$, $\mathbf{D}$ is a positive definite symmetric matrix of dimension $qN \times qN$ and $|\mathbf{D}| = \det \mathbf{D}$.

The proof that $h_Y(\mathbf{y})$ is an honest probability function (i.e. $\int_{-\infty}^{\infty} h_Y(\mathbf{y}) d\mathbf{y} = 1$) is trivial and we shall not present it here.

It can be also easily shown that

$$E(\mathbf{Y}) = \mathbf{U}, \quad \mathbf{D}^2(\mathbf{Y}) = \mathbf{D}$$

Definition 1.5. The characteristic function of a random matrix $\mathbf{Y}$ is defined as

$$\phi_Y(\mathbf{T}) = E(e^{i \text{tr} \mathbf{T}' \mathbf{Y}}), \quad i = \sqrt{-1}, \quad \mathbf{T} \in R^{q \times N}$$

Using some orthogonal transformations it can be shown that if $\mathbf{Y}$ is $N(\mathbf{U}, \mathbf{D})$, then

$$\phi_Y(\mathbf{T}) = e^{i \text{tr} (\mathbf{T}' \mathbf{U} - \frac{1}{2} \mathbf{D} \mathbf{W}(\mathbf{T}))}$$

By means of the characteristic function, the following properties are easily verified:

P.3. If $\mathbf{Y}' = (Y_1, \dots, Y_q)$ is a random vector $N(\mathbf{U}, \Sigma)$ and $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_q)$ is a constant vector then $\mathbf{Y}\mathbf{A}$ is distributed $N(\mathbf{U}\mathbf{A}, \Sigma \otimes \mathbf{A}'\mathbf{A})$.

P.4. If $\mathbf{Y}$ is a $q \times N$ random matrix distributed $N(\mathbf{U}, \Sigma \otimes \Theta)$, Σ and Θ being positive definite of order q and N , respectively, then, if the matrix product $\mathbf{C}\mathbf{Y}\mathbf{G}$ is well defined, we have that $\mathbf{C}\mathbf{Y}\mathbf{G} + \mathbf{D}$ is $N(\mathbf{C}\mathbf{U}\mathbf{G} + \mathbf{D}, \mathbf{C}\Sigma\mathbf{C}' \otimes \mathbf{G}'\Theta\mathbf{G})$ distributed

P.5. If $\mathbf{Y}$ is a $q \times N$ random matrix distributed $N(\mathbf{U}, \Sigma \otimes \Theta)$ with Σ and Θ defined as before, then the matrix

$\Sigma^{-1/2}(\mathbf{Y} - \mathbf{U})\Theta^{-1/2}$ is distributed $N(\mathbf{0}, \mathbf{I}_q \otimes \mathbf{I}_N)$. Here, with

$\mathbf{A}^{-1/2}$ we have denoted the nonsingular matrix with the property $\mathbf{A}^{-1/2}\mathbf{A}\mathbf{A}^{-1/2} = \mathbf{I}$

P.6. Let $\mathbf{Y}^\alpha = (Y_{1\alpha}, \dots, Y_{q\alpha})$, $1 \leq \alpha \leq N$, be a sample of size N from a normal distribution with mean $\mathbf{U}$ and positive definite covariance matrix Σ . The Likelihood of the sample observation $\mathbf{Y}^\alpha$, $1 \leq \alpha \leq N$, is distributed $N(\mathbf{U}, \Sigma \otimes \mathbf{I}_N)$, where $\mathbf{U} = (\underbrace{\mathbf{U}, \dots, \mathbf{U}}_{N\text{-times}})$.

P.7. If $\mathbf{D} \in R^{q \times N}$ can be written in the form $\Sigma \otimes \Theta$, $\Sigma \in R^{q \times q}$ $\Theta \in R^{N \times N}$ then $\text{tr } \mathbf{D}^{-1}\mathbf{W}(\mathbf{Y}) = \text{tr } \Sigma^{-1}\mathbf{Y}\Theta^{-1}\mathbf{Y}' = \text{tr } \Theta^{-1}\mathbf{Y}'\Sigma^{-1}\mathbf{Y}$, where $\mathbf{Y}$ is a $q \times N$ matrix.

Definition 1.5 : Let $\mathbf{Y} \in R^{q \times N}$ be a random matrix distributed $N(\mathbf{0}, \mathbf{D})$ and $n\mathbf{S} = \sum_{i=1}^n \mathbf{W}(\mathbf{Y}_i)$ a matrix independent of $\mathbf{Y}$, where $\mathbf{Y}_i$, $1 \leq i \leq n$ are random matrices each one distributed $N(\mathbf{0}, \mathbf{D})$. Hotelling's generalized $\mathbf{T}^2$ statistic is defined by

$$\mathbf{T}^2 = \text{tr } \mathbf{S}\mathbf{W}(\mathbf{Y}) = \sum_{i=1}^n \text{tr } \mathbf{W}(\mathbf{Y}_i) \mathbf{W}(\mathbf{Y})$$

Definition 1.6 : ([4]) : The $qN \times qN$ random variable $\mathbf{W}(\mathbf{Y})$ is said to have a $qN \times qN$ dimensional Wishart distribution if its probability density function is given by

$$f(\mathbf{W}(\mathbf{y})) = \frac{|\mathbf{W}(\mathbf{y})|^{\frac{1}{2}(n-qN-1)}}{2^{nqN/2} |\mathbf{D}|^{n/2} \Gamma_{qN}\left(\frac{n}{2}\right)} e^{-\frac{1}{2} \text{tr } \mathbf{W}(\mathbf{y}) \mathbf{D}^{-1}} \quad (1.3)$$

where $\Gamma_m(a)$ is the multivariate gama function defined by

$$\Gamma_m(a) = \int_{\mathbf{S} > 0} e^{-\text{tr } \mathbf{S}} |\mathbf{S}|^{a - \frac{1}{2}(m+1)} d\mathbf{S} = \pi^{\frac{m(m+1)}{4}} \left[\prod_{i=1}^m \Gamma\left(a - \frac{1}{2}(i-1)\right) \right];$$

$\mathbf{D}$ is a $qN \times qN$ positive definite matrix; n is a given natural number defined to be the degrees of freedom and $\mathbf{W}(\mathbf{Y})$ is the matrix of matrices defined by the expresion (1.1), where $\mathbf{Y}$ is a $q \times N$ random matrix.

We shall denote by $W(qN, n, \mathbf{D})$ the set of $(qN)^2$ -dimensional random matrices whose probability density function is given by (1.3.). It is easy to prove (see [1]) that the characteristic function of a matrix distributed $W(qN, n, \mathbf{D})$ is given by the expression

$$\phi_{W(Y)}(\mathbf{W}(\mathbf{T})) = \frac{|\mathbf{D}|^{n/2}}{|\mathbf{D}^{-1} - 2i\mathbf{W}(\mathbf{T})|^{n/2}}$$

Like in the more particular definition of the Wishart random matrix, (see, for instance, [4]), it is possible to make the transformation $\mathbf{W}(\mathbf{Y}) = \mathbf{C}\mathbf{Z}$, where $\mathbf{C}$ is a nonsingular matrix such

that $\mathbf{C}\mathbf{D}\mathbf{C}' = \mathbf{I}_{qN}$, $\mathbf{C}\mathbf{W}(\mathbf{T})\mathbf{C}' = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix}$, in order to prove the following lemma

Lemma 1.1: Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n, \mathbf{Z}_i \in R^{qN}$, $1 \leq i \leq n$, ($n > qN$), n independent random matrices distributed $N(\mathbf{0}, \mathbf{D})$. The random matrix $\sum_{i=1}^n \mathbf{W}(\mathbf{Z}_i)$ is distributed $W(qN, n, \mathbf{D})$.

Definition 1.7. [5]: The random matrix

$$\mathbf{F} = \mathbf{W}(\mathbf{Y}_1)^{1/2} \mathbf{W}(\mathbf{Y}_2)^{-1} \mathbf{W}(\mathbf{Y}_1)^{1/2}$$

where $\mathbf{W}(\mathbf{Y}_1)$ and $\mathbf{W}(\mathbf{Y}_2)$ are independent distributed $W(qN, n, \mathbf{D})$ and $W(q, N, n_2, \mathbf{D})$ respectively is called $\mathbf{F}$ random matrix, and its distribution function, F multivariate distribution function.

§2. The multivariate classical linear model

Consider the classical Multivariate Linear Model

$$\mathbf{Y} = \beta\mathbf{X} + \mathbf{e} \quad (2.1)$$

where

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ \vdots & \vdots & & \vdots \\ Y_{q1} & Y_{q2} & \cdots & Y_{qN} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_q \end{bmatrix}$$

is a matrix of qN observation :

$$\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_N) = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & & \vdots \\ X_{p1} & X_{p2} & \cdots & X_{pN} \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_p \end{bmatrix}$$

is a known matrix and

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_q \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \vdots & & & \\ \beta_{q1} & \beta_{q2} & \cdots & \beta_{qp} \end{bmatrix}$$

$p \leq N$, is a matrix of unknown parameters. The problem is to estimate β under the hypothesis

$$\Omega: \begin{cases} E(\mathbf{e}) = \mathbf{0} \\ D^2(\mathbf{e}) = \Sigma \otimes \mathbf{I}_N \end{cases} \quad (2.2)$$

where Σ is a $q \times q$ positive definite matrix and $\mathbf{I}_N$ is the $N \times N$ unit matrix.

The more general problem

$$\tilde{\mathbf{Y}} = \beta \tilde{\mathbf{X}} + \tilde{\mathbf{e}}$$

$$\Omega: \begin{cases} E(\tilde{\mathbf{e}}) = \mathbf{0} \\ D^2(\tilde{\mathbf{e}}) = \Sigma \otimes \mathbf{B} \end{cases}$$

where $\mathbf{B}$, $|\mathbf{B}| > 0$ is a known $N \times N$ matrix, can be reduced to (2.1) under the hypothesis (2.2) by considering the transformation

$$\mathbf{Y} = \tilde{\mathbf{Y}}\mathbf{C}$$

where $\mathbf{C}$, $|\mathbf{C}| > 0$ is the matrix with the property $\mathbf{C}\mathbf{B}'\mathbf{C} = \mathbf{I}_N$. In fact in such a case we have

$$\tilde{\mathbf{Y}}\mathbf{C} = \beta \tilde{\mathbf{X}}\mathbf{C} + \tilde{\mathbf{e}}\mathbf{C},$$

and putting

$$\mathbf{X} = \tilde{\mathbf{X}}\mathbf{C}, \quad \mathbf{e} = \tilde{\mathbf{e}}\mathbf{C}$$

we get (1.1). Furthermore, we have

$$E(\tilde{\mathbf{e}}\mathbf{C}) = \mathbf{0}$$

$D^2(\tilde{\mathbf{e}}\mathbf{C}) = D^2(\mathbf{C}'\mathbf{e}') = \Sigma \otimes \mathbf{I}_N$ (property P.2), so, the hypothesis (2.2) is also obtained.

§3. Normal equations and least square estimators

The equation in β

$$\mathbf{X}\mathbf{X}'\beta = \mathbf{X}\mathbf{Y}. \quad (3.1)$$

obtained by differentiating (to minimize) the function

$$\|Y - \beta X\|^2 = \text{tr}(Y - \beta X)(Y - \beta X)' \quad (3.2)$$

is called Normal Equations System. The methods of obtaining β by minimizing (3.2) is called the *Least Square Method*.

Denote by $\mu(\tilde{\beta})$ the set of matrices β which satisfies (3.1) and by $\mu(\hat{\beta})$ the set of β which minimize (3.2). It can be easily verified that $\mu(\tilde{\beta}) \neq \emptyset$ and $\mu(\hat{\beta}) \neq \emptyset$. Furthermore, $\mu(\tilde{\beta}) = \mu(\hat{\beta})$, and therefore we can set the following lemma:

Lemma 3.1 : *For the model (2.1) always exists a matrix-least square estimator (MLSE); any of these estimators satisfies (3.1) and every solution of (3.1) (which does not depend on Y) is a MLSE of β .*

Lemma 3.2. :

$$(Y - \beta X)(Y - \beta X)' = YY' - YX'\beta$$

Proof: In fact

$$(Y - \beta X)(Y - \beta X)' = YY' - YX'\beta - \beta(XY' - XX'\beta) = YY' - YX'\beta' \quad (\text{Q.E.D.})$$

If $\text{rank } X = r < p$, the system (3.1) has the solution

$$\hat{\beta} = (XX')^{-}XY'$$

where $(XX')^{-}$ is the generalized inverse of XX' . If $r = p$, then $\text{rank}(XX') = p$ and $(XX')^{-} = (XX')^{-1}$ is the classical inverse.

§4. Estimable matrix functions

Definition 4.1 : We shall say that the function

$$\psi = C\beta' ; C = (C_{ij}) \in R^{q \times p}, (C \text{ known}),$$

is an estimable matrix function if there exists $A = (a_{ik}) \in R^{q \times N}$ such that

$$E(AY') = \psi$$

It is easy to see that the following conditions are equivalent :

- i. ψ is an estimable matrix function
- ii : There exists $A \in R^{q \times N}$ such that $C = AX'$
- iii. There exists $\hat{\psi} = A^*Y' = (a_1^*, \dots, a_i^*, \dots, a_q^*)Y'$, $E(\hat{\psi}) = \psi$, such that $\hat{\psi}$ is unique for all solutions $\hat{\beta}$ of the normal system. (In this case, given $A_{q \times N} = (a_1, \dots, a_i, \dots, a_q)$,

a_i^* denotes the projection of a_i on $\varepsilon(X')$ the vector space generated by the columns of X' .

§5. Multivariate Gauss-Markov theorem

First of all we shall prove two preliminary lemmas :

Lemma 5.1.

$$\text{a. } |AA' - A^*A^{*'}| > 0$$

$$\text{b. } |AA'| > |A^*A^{*'}|$$

and the equality holds iff $A = A^*$

Proof. a. From the fact that

$$AA^{*'} = A^{*'}A = A^*A^{*'}$$

We get

$$|AA' - A^*A^{*'}| = |(A - A^*)(A - A^*)'| > 0$$

where the equality holds iff $A = A^*$

b. Let us consider the case $|A^*A^{*'}| > 0$. Then it is known [1] that if $|AA'| > 0$, then there exists F , $|F| > 0$, such that

$$F'AA'F = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_q \end{bmatrix}, F'A^*A^{*'}F = I_q$$

where d_i , $1 \leq i \leq q$, are the latent roots of the characteristic equation

$$|AA' - dA^*A^{*'}| = 0$$

Thus, we have

$$\frac{|F'AA'F|}{|F'A^*A^{*'}F|} = d_1 d_2 \cdots d_q = \frac{|AA'|}{|A^*A^{*'}|}$$

but from $|AA'| > 0$, follows $d_1 d_2 \cdots d_q > 0$, so $|AA'| > |A^*A^{*'}|$, where the equality holds iff $d_1 = d_2 = \cdots = d_q = 1$, i.e., $A = A^*$.

The case $|AA'| = 0$ is trivial because in this case $|A^*A^{*'}| = 0$.

(Q.E.D.)

Lemma 5.2 : $(Y'_i - X'\hat{\beta}'_i) \perp \varepsilon(X') \ (\forall) \ i, 1 \leq i \leq q$

Proof. We can see that the matrix $X'\hat{\beta}_i$ can be written in the form

$$X'\hat{\beta}_i = \sum_{j=1}^p \beta_{ij} \varepsilon_j$$

i.e., $\mathbf{X}'\boldsymbol{\beta}_i \in \varepsilon(\mathbf{X}')$. If we denote by $\hat{\mathbf{Z}}_i$ the projection of the vector $\mathbf{Y}'_i$ on $\varepsilon(\mathbf{X}')$, then it follows that $\hat{\mathbf{Z}}_i = \sum_{j=1}^p \hat{\boldsymbol{\beta}}_{ij} \boldsymbol{\varepsilon}'_j$, where $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_{ij})_{1 \leq i \leq q, 1 \leq j \leq p}$ is the MLSE of $\boldsymbol{\beta}$. From $\mathbf{X}'\boldsymbol{\beta}'_i \in \varepsilon(\mathbf{X}')$ it follows that

$$(\mathbf{Y}'_i - \mathbf{X}'\hat{\boldsymbol{\beta}}'_i) \perp \varepsilon(\mathbf{X}')$$

(Q.E.D.)

Theorem 5.1. (Gauss-Markov). Under the hypothesis (2.2), the MLSE $\hat{\boldsymbol{\beta}}$ defined in (3.3) is best linear unbiased in the following sense: If with $\boldsymbol{\mu}(\tilde{\boldsymbol{\psi}})$ denote the class of linear unbiased estimators of the estimable matrix — function $\boldsymbol{\psi} = \mathbf{C}\boldsymbol{\beta}$, then, for every $\boldsymbol{\psi}$ there exists $\hat{\boldsymbol{\psi}}^ \in \boldsymbol{\mu}(\boldsymbol{\psi})$ such that*

$$|D^2(\hat{\boldsymbol{\psi}}^*)| < |D^2(\tilde{\boldsymbol{\psi}})| \quad (\forall) \tilde{\boldsymbol{\psi}} \in \boldsymbol{\mu}(\tilde{\boldsymbol{\psi}}), \quad \tilde{\boldsymbol{\psi}} \neq \hat{\boldsymbol{\psi}}^* ;$$

furthermore, $\hat{\boldsymbol{\psi}}^* = \hat{\boldsymbol{\psi}} = \mathbf{C}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is the MLSE of $\boldsymbol{\beta}$.

Proof. According to iii, (§3), there exists $\hat{\boldsymbol{\psi}} \in \boldsymbol{\mu}(\tilde{\boldsymbol{\psi}})$,

$$\hat{\boldsymbol{\psi}} = \mathbf{A}^* \mathbf{Y}' = (a_1^*, \dots, a_q^*) \mathbf{Y}'$$

Taking into account definition 1.3 and property P.2 we have

$$D^2(\mathbf{A}\mathbf{Y}') = \mathbf{A}\mathbf{A}' \otimes \boldsymbol{\Sigma}, \quad D^2(\mathbf{A}^* \mathbf{Y}') = \mathbf{A}^{**} \mathbf{A}' \otimes \boldsymbol{\Sigma}$$

and from (b), lemma 4.1 we get

$$|D^2(\mathbf{A}\mathbf{Y}')| = |\mathbf{A}\mathbf{A}'|^q |\boldsymbol{\Sigma}|^q \geq |\mathbf{A}^{**} \mathbf{A}'|^q |\boldsymbol{\Sigma}|^q = |D^2(\mathbf{A}^* \mathbf{Y}')|$$

where the equality holds iff $\mathbf{A} = \mathbf{A}^*$.

To prove that $\hat{\boldsymbol{\psi}} \in \boldsymbol{\mu}(\hat{\boldsymbol{\psi}})$, we shall use the following statements:

$$\mathbf{C} = \mathbf{A}\mathbf{X}' \quad (\text{ii, §3})$$

$$a_i^* \in \varepsilon(\mathbf{X}')$$

$$(\mathbf{Y}'_i - \mathbf{X}'\hat{\boldsymbol{\beta}}'_i) \perp \varepsilon(\mathbf{X}'), \quad (\forall) i, 1 \leq i \leq q \quad (\text{Lemma 4.2.})$$

From this we get $\mathbf{A}^*(\mathbf{Y}' - \mathbf{X}'\hat{\boldsymbol{\beta}}') = 0$, as

$$\hat{\boldsymbol{\psi}} = \mathbf{A}^* \mathbf{Y}' = \mathbf{A}^* \mathbf{X}'\hat{\boldsymbol{\beta}}' = \mathbf{C}\hat{\boldsymbol{\beta}}'$$

(Q.E.D.)

§6. Some remarks on the estimable matrix functions

Let us suppose now that $\mathbf{Y}$ is distributed $N(\beta\mathbf{X}, \Sigma \otimes \mathbf{I}_N)$. Let $\hat{\mathbf{A}}$ be the matrix such that $E(\hat{\mathbf{A}}\mathbf{Y}) = E(\hat{\psi}) = \psi$. Then, from properties P.2 and P.4 it follows that $\hat{\psi}$ is distributed $N(\psi, \hat{\mathbf{A}}\hat{\mathbf{A}}' \otimes \Sigma)$ and $(\hat{\psi} - \psi)$ is distributed $N(\mathbf{0}, \hat{\mathbf{A}}\hat{\mathbf{A}}' \otimes \Sigma)$. Property P.5 implies that $(\hat{\mathbf{A}}\hat{\mathbf{A}}' \otimes \Sigma)^{-1/2} (\hat{\psi} - \psi)$ is $N(\mathbf{0}, \mathbf{I}_q \otimes \mathbf{I}_N)$ distributed (i.e., $\text{tr} (\hat{\mathbf{A}}\hat{\mathbf{A}}')^{-1} (\hat{\psi} - \psi) \Sigma^{-1} (\hat{\psi} - \psi)'$ is χ^2 distributed with qN degrees of freedom), and definition 1.6 implies that $\mathbf{S}_1 = (\hat{\mathbf{A}}\hat{\mathbf{A}}')^{-1} (\hat{\psi} - \psi)' \Sigma^{-1} (\hat{\psi} - \psi)$ is $W(q, q, \mathbf{I}_q)$ distributed.

Let
$$\mathbf{S} = \frac{(\mathbf{Y}' - \mathbf{X}'\hat{\beta}')' (\mathbf{Y}' - \mathbf{X}'\hat{\beta}')}{N - r}$$

be an unbiased linear estimate of Σ . By lemma 1.1. and from the possible decomposition $(N-r) \mathbf{S} = \sum_{i=r}^N \mathbf{Z}_i \mathbf{Z}_i'$, where $\mathbf{Z}_i$ is distributed $N(\mathbf{0}, \Sigma)$, it follows that $\mathbf{S}_2 = (N-r) \mathbf{S}$ is distributed $W(q, N-r, \Sigma)$, thus, by definition 1.7 it follows that the random matrix $\mathbf{S}_1^{-1/2} \mathbf{S}_2^{-1} \mathbf{S}_1^{-1/2}$ is multivariate F distributed with q and $N-r$ degrees of freedom. Confidence regions and tests for hypothesis of the form $\psi_1 = \psi_q$ are obtained immediately.

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Instituto Tecnológico Regional de Cd. Juárez
(Mexico)

SUR UN THÉORÈME DE COMPARABILITÉ

N. MICU

Soit $\mathcal{F}$ la famille des fonctions aleatoires finies sur l'espace probabilisé $(\Omega, \mathcal{H}, P)$ telles que $X_t \rightarrow +\infty$ ($t \rightarrow \infty$) en probabilité

Si $\alpha = \{X_t\} \in \mathcal{F}$ $a \in (0, 1)$ et X est une variable aleatoires finie notons

$$N_\alpha(X, a) = \sup \{t : P(X_t \leq X) \geq a\}$$

Définition 1: $\alpha = \{X_t\} \in \mathcal{F}$ jouit de la propriété (M) si

$$\lim_{t \rightarrow \infty} \frac{1}{t} N_\alpha(X_t, a) = 1 \quad \forall a \in (0, 1)$$

Définition 2: $\alpha = \{X_t\}, \alpha' = \{X'_t\}$ de $\mathcal{F}$ sont comparables s'il existe $\rho(\alpha, \alpha') \in (0, \infty)$ tel que

$$\lim_{t \rightarrow \infty} \frac{N_\alpha(X_t, a)}{N_{\alpha'}(X_t, a)} = \rho(\alpha, \alpha'); \quad \lim_{t \rightarrow \infty} \frac{N_\alpha(X'_t, a)}{N_{\alpha'}(X'_t, a)} = \rho(\alpha, \alpha') \quad \forall a \in (0, 1)$$

On montre que si α et α' jouissent de la propriété (M) alors les deux conditions de la définition 2 sont équivalentes.

On montre aussi ([1]) qu'il existe une constante $\rho \in (0, \infty)$ telle que si $t \rightarrow \infty, t' \rightarrow \infty, \frac{t}{t'} \rightarrow \lambda$ alors

$$\lambda > \rho \Rightarrow P(X_t > X'_{t'}) \rightarrow 1 (t \rightarrow \infty)$$

$$\lambda < \rho \Rightarrow P(X_t < X'_{t'}) \rightarrow 1 (t \rightarrow \infty)$$

Lemme: Si (a_n) et (b_n) sont des suites de réels la première nondécroissante et la seconde (strictement) croissante et

$$a_n \rightarrow \infty, b_n \rightarrow \infty, \frac{a_n}{b_n} \rightarrow \lambda \in (0, \infty), a_n - a_{n-1} \leq b_n - b_{n-1} \quad (n \geq 1)$$

$(a_0 = b_0 = 0)$ alors il existe une partie borélienne A de R_+ telle que

$$\mu(A \cap (0, b_n)) = a_n; \quad \lim_{x \rightarrow \infty} \frac{1}{x} \mu(A \cap (0, x)) = \lambda$$

(μ = mesure de Lebesgue).

Démonstration: La fonction

$$\varphi(t) = a_n + (t - b_n) \cdot \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \quad \text{pour } b_n \leq t < b_{n+1}$$

est continue et croissante. On voit que

$$\varphi(b_n) = a_n; \quad \varphi(t) - \varphi(t') \leq t - t' \quad (0 \leq t' < t)$$

Pour tout $t > 0$ soit $n = n(t)$ tel que $b_n \leq t < b_{n+1}$

Dans ce cas

$$\frac{a_n}{b_n} - \frac{\varphi(t)}{t} = \frac{b_{n+1}(t - b_n)}{t(b_{n+1} - b_n)} \cdot \left(\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} \right)$$

et donc

$$\left| \frac{a_n}{b_n} - \frac{\varphi(t)}{t} \right| \leq \left| \frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} \right|$$

Par conséquent

$$\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} = \lambda$$

Soit (t_n) une suite croissante de nombres réels telle que (b_n) soit une sous-suite de (t_n) et

$$\lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 1 \quad \left(\Rightarrow \lim_{n \rightarrow \infty} \frac{\varphi(t_{n+1})}{\varphi(t_n)} = 1 \right)$$

Soit aussi une suite d'intervalles (I_k) tels que

$$I_k \subset [t_k, t_{k+1}); \quad \mu(I_k) = \varphi(t_k) - \varphi(t_{k-1}) \quad (t_0 = 0)$$

On note

$$A = \bigcup_{k=1}^{\infty} I_k$$

On voit que pour tout $s > 1$

$$\mu(A \cap (0, t_s)) = \sum_{k=1}^s \mu(I_k) = \varphi(t_s)$$

d'où

$$\mu(A \cap (0, b_n)) = \varphi(b_n) = a_n$$

Pour tout $t > 0$ soit $s = s(t)$ telle que $t \in [t_{s-1}, t_s)$. Dans ce cas

$$\mu(A \cap (0, t_{s-1})) \leq \mu(A \cap (0, t)) \leq \mu(A \cap (0, t_s))$$

et donc

$$\frac{t_{s-1}}{t_s} \cdot \frac{\varphi(t_{s-1})}{t_{s-1}} \leq \frac{\mu(A \cap (0, t))}{t} \leq \frac{\varphi(t_s)}{t_s} \cdot \frac{t_s}{t_{s-1}}$$

Il en résulte

$$\lim_{t \rightarrow \infty} \frac{\mu(A \cap (0, t))}{t} = \lambda.$$

Soit $\beta = \{X_t\}$, $\beta' = \{X'_t\}$ des fonction aléatoires mesurables à trajectoires intégrables sur tout intervalle borné de R_+ . Pour tout t les applications de $\omega \in \Omega$

$$Y_t = \int_{(0,t)} X(u) du; \quad Y'_t = \int_{(0,t)} X'(u) du$$

sont $\mathcal{H}$ mesurables. Pour toute partie borélienne de R_+ on considère la fonction aléatoire $Z(t) = Z(t, A, \beta, \beta')$

$$Z(t) = \begin{cases} X(u) & \text{pour } t \in A, \mu(A \cap (0, t)) = u \\ -X'(u) & \text{pour } t \in A^c, \mu(A^c \cap (0, t)) = v \end{cases}$$

Théorème: Si $\alpha = \{Y_t\}$, $\alpha' = \{Y'_t\} \in \mathcal{F}$ alors elles jouissent de la propriété (M) sont comparables et $\rho(\alpha, \alpha') = \rho$ si et seulement si pour tout borélien $A \subset (0, +\infty)$ tel que

$$\lim_{x \rightarrow \infty} \frac{1}{x} \mu(A \cap (0, x)) = \lambda$$

on a

$$\lambda > \frac{\rho}{1 + \rho} \Rightarrow P\left(\int_{(0,x)} Z(t)dt > 0\right) \rightarrow 1 \quad (x \rightarrow \infty)$$

$$\lambda < \frac{\rho}{1 + \rho} \Rightarrow P\left(\int_{(0,x)} Z(t)dt < 0\right) \rightarrow 1 \quad (x \rightarrow \infty)$$

Démonstration: Soit $A_t = A \cap (0, t)$, $v(t) = \mu(A_t)$, ($t > 0$)
On montre que pour tout borelien $B \cap (0, v(t))$ on a

$$\mu(v^{-1}(B) \cap A) = \mu(B)$$

et donc

$$\int_{(0, v(t))} X(u) du = \int_{A \cap (0, t)} X(v(u)) du = \int_{A \cap (0, t)} Z(u) du$$

De la même manière on montre que

$$\int_{(0, t-v(t))} X'(u) du = - \int_{A^c \cap (0, t)} Z(u) du$$

De les deux dernières relations on obtient

$$\int_{(0, t)} Z(u) du = \int_{(0, v(t))} X(u) du - \int_{(0, t-v(t))} X'(u) du$$

Supposons que α et α' jouissent de la propriété (M), sont comparables
et $\rho(\alpha, \alpha') = \rho$. Si $\lambda > \rho \frac{\rho}{1 + \rho}$ alors

$$\lim_{x \rightarrow \infty} \frac{v(x)}{x - v(x)} = \frac{\lambda}{1 - \lambda} > \rho$$

et donc pour $x \rightarrow \infty$ on a

$$P\left(\int_{(0, x)} Z(t) dt > 0\right) = P(Y(v(x)) > Y'(x - v(x))) \rightarrow 1$$

Semblablement il résulte

$$\lambda < \frac{\rho}{1 + \rho} \Rightarrow P\left(\int_{(0, x)} Z(t) dt < 0\right) \rightarrow 1 \quad (x \rightarrow \infty)$$

Réciproquement, supposons les condition (1) vérifiées.

Soit (a_n) et (b_n) deux suites de nombres réels positifs $a_n \rightarrow \infty$, $b_n \rightarrow \infty$,

$\frac{a_n}{b_n} \rightarrow \lambda$ et A un sous-ensemble borélien de R tel que

$$\mu(A \cap (0, a_n + b_n)) = a_n$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mu(A \cap (0, t)) = \frac{\lambda}{1 + \lambda}$$

Si $\lambda > \rho$ alors $\frac{\lambda}{1+\lambda} > \frac{\rho}{1+\rho}$ et donc

$$P(Y(a_n) > Y'(b_n)) = P\left(\int_{(0, a_n + b_n)} Z(t) \, dt > 0\right) \rightarrow 1 \quad (n \rightarrow \infty)$$

D'une manière analogue on montre que

$$\lambda < \rho \Rightarrow P(Y(a_n) > Y'(b_n)) \rightarrow 1 \quad (n \rightarrow \infty)$$

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THE MARKOVIAN CHARACTER OF SOME MULTI-SERVER QUEUEING SYSTEMS

EUGEN RADIAN

The aim of this paper is to prove the Markovian character of the queueing systems when the customers arrive individually with independent interarrival times, having the same distribution F and being served in several service stations. The service times are independent with the same negative exponential distribution with parameter λ . Moreover, the interarrival times and the service times are independent.

The service process of the customers being at the beginning of the service in the system (if we require not to be any later arrivals) is a death process. If we use the strong Markov property for this process we find that the service process with only one arrival after its beginning, regarded to start from this arrival time, is of the same type with the first one. This fact permits us to prove that "the number of customers in system at the arrival times" is a Markov chain.

§ 1. Some Preliminary Facts

Lemma 1.1: Let $\{E, \mathcal{K}, P\}$ be a probability space and $\beta_1, \dots, \beta_n, \dots$ a sequence of independent, $(0, \infty)$ -valued, random variables on it, having the distributions

$e_{\lambda_1}, \dots, e_{\lambda_n}, \dots$. Let $i \leq k$ be natural numbers.

Then the following random variables

$$\beta_i, \beta_1 - \beta_i, \dots, \beta_{i-1} - \beta_i, \beta_{i+1} - \beta_i, \dots, \beta_k - \beta_i, \beta_{k+1}, \dots, \beta_n, \dots$$

are independent and their distributions are:

$$e_{\lambda_1 + \dots + \lambda_k}, e_{\lambda_1}, \dots, e_{\lambda_{i-1}}, e_{\lambda_{i+1}}, \dots, e_{\lambda_k}, e_{\lambda_{k+1}}, \dots, e_{\lambda_n}, \dots,$$

with respect to $P_{(\beta_i = \min(\beta_1, \dots, \beta_k))}$

Proof: See [3].

Lemma 2.1: Let $\{E, \mathcal{K}, P\}$ be a probability space and

$A_1, \dots, A_n$; $A_i \in \mathcal{K}$, $i = 1, \dots, n$, $B \in \mathcal{K}$. If $P_{A_1}(B) = \dots = P_{A_n}(B)$ and

$$A_i \cap A_j = \emptyset, \quad i \neq j, \quad \text{then} \quad P_{A_1 \cup \dots \cup A_n}(B) = P_{A_1}(B) = \dots = P_{A_n}(B).$$

Proof: See [3].

Proposition 1.1: Let I be a denumerable set. Let $\{E^i, \mathcal{K}^i, P^i\}$ be a probability space for every $i \in I$ and $y_n^i: E^i \rightarrow I$ be a random variable for every $n = 0, 1, \dots$. Let us suppose:

a. $y_0^i = i$ P^i a.e.

b. the distribution of $(y_1^i, y_2^i, \dots)$ with respect to $P^{(y_1^i=j)}$ is the same with the distribution of $(y_0^j, y_1^j, \dots)$ with respect to P^j .

Then, if we denote by $q_{ij} = P^i(y_1^i = j)$, $Q = (q_{ij})_{i,j \in I}$, every $(y_n^i)_{n=0,1,\dots}$ is a homogeneous Markov chain which corresponds to the stochastic matrix Q .
Proof: See [3].

Lemma 3.1.: Let I be a denumerable set. Let $\{E^i, \mathcal{K}^i, P^i\}$ be a probability space for every $i \in I$ and $y_n^i: E^i \rightarrow I$, $\sigma_n^i: E^i \rightarrow (0, \infty)$, $n = 0, 1, \dots$ be random variables with the following properties:

a. $y_0^i = i$ P^i a.e.

b. σ_0^i has the distribution e_{λ_i} ($\lambda_i \in (0, \infty)$) with respect to P^i .

c. σ_0^i and $\mathcal{B}(y_1^i, y_2^i, \dots, \sigma_1^i, \sigma_2^i, \dots)$ are independent.

d. the distribution of $(y_1^i, y_2^i, \dots, \sigma_1^i, \sigma_2^i, \dots)$ with respect to $P^{(y_1^i=j)}$ is the same with the distribution of $(y_0^j, y_1^j, \dots, \sigma_0^j, \sigma_1^j, \dots)$ with respect to P^j .

Then if we denote $q_{ij} = P^i(y_1^i = j)$ and if we define:

$$x_t^i = y_0^i \text{ if } t < \sigma_0^i$$

$$x_t^i = y_n^i \text{ if } \sigma_0^i + \dots + \sigma_{n-1}^i < t \leq \sigma_0^i + \dots + \sigma_n^i$$

$$x_t^i = \delta \text{ if } t \geq \sigma_0^i + \dots + \sigma_n^i + \dots$$

we obtain a homogeneous Markov process $(x_t^i)_{t \in [0, \infty)}$ on $\{E^i, \mathcal{K}^i, P^i\}$ for every $i \in I$, corresponding to the minimal semigroup given by $Q = (q_{jk})_{j,k \in I}$ and $\Lambda = (\lambda_i)_{i \in I}$

Proof: See [3]

Lemma 4.1.: Let $\{E, \mathcal{K}, P\}$ be a probability space. If $(\beta_t)_{t \in [0, \infty)}$ is a (Q_t) process on $\{E, \mathcal{K}, P\}$, then $(\beta_t)_{t \in [0, \infty)}$ is a (Q_t) process with respect to $P_{(\beta_0=i)}$.

Proof: It is enough to show that $M_P(f|\mathcal{F}) = M_{P_A}(f|\mathcal{F})$ if $A \in \mathcal{F}$ ($\mathcal{F}$ being a borelian field on E , $\mathcal{F} \subset \mathcal{K}$)

Let $B \in \mathcal{F}$. We have:

$$\int_B f dP_A = \frac{1}{P(A)} \int_{A \cap B} f dP = \frac{1}{P(A)} \int_{A \cap B} M_P(f|\mathcal{F}) dP = \int_B M_P(f|\mathcal{F}) dP_A$$

Lemma 5.1.: Let $\{E, \mathcal{F}, Q\}$ be a probability space and $\mathcal{G}, \mathcal{H}$, borelian fields on E , $\mathcal{G} \subset \mathcal{F}$, $\mathcal{H} \subset \mathcal{F}$. Let $f: E \rightarrow \mathbb{R}$ be a random variable. Let us suppose that $\mathcal{G}$ and $\mathcal{B}(\mathcal{B}(f) \cup \mathcal{H})$ are independent. Then $M(f|\mathcal{H}) = M(f|\mathcal{B}(\mathcal{H} \cup \mathcal{G}))$.

Proof: It is enough to show that $\mathcal{G}, \mathcal{H}, \mathcal{B}(f)$ is a Markov triplet. Hence it is enough to prove that $\mathcal{B}(f), \mathcal{H}, \mathcal{G}$ is a Markov triplet, i.e. for every $A \in \mathcal{G}$ $M(\chi_A | \mathcal{B}(\mathcal{B}(f) \cup \mathcal{H})) = M(\chi_A | \mathcal{H})$, which is true because the both parts of this equality are $P(A) | \mathcal{G}$ and $\mathcal{B}(\mathcal{B}(f) \cup \mathcal{H})$ are independent).

Remark 1.1.: Let $\{E, \mathcal{H}, P\}$ be a probability space and $\mathcal{H} \supset \mathcal{F}_1, \dots, \mathcal{F}_n$, ... be borelian fields, independent with respect to P .

Then they are also independent with respect to P_A $A \in \mathcal{F}_1$ and $P_A | \mathcal{F}_i = P | \mathcal{F}_i$ for every $i \geq 2$.

§ 2. The Queueing System G/M/2

Remark 1.2: We are going to deal with the queue G/M/2, when the customers arrive individually, are served in order of their arrival and they can use two servers.

Let $\{E, \mathcal{H}, P\}$ be a probability space and let be on it the following positive-valued random variables:

$\tau_1, \dots, \tau_n, \dots$ the service times, independent and identically distributed with the distribution e_λ ($\lambda > 0$) random variables (τ_i is the i th service time)

$\sigma_1, \dots, \sigma_n, \dots$ the interarrival times, independent and identically distributed with the distribution F random variables.

Let us suppose that $\mathcal{B}(\tau_1, \dots, \tau_n, \dots)$ and $\mathcal{B}(\sigma_1, \dots, \sigma_n, \dots)$ are independent.

Let $(\beta_i^t)_{t \in [0, \infty)}$ be the stochastic process defined by:

$$\beta_i^1(\tau_1) = \begin{cases} 1 & \text{if } t < \tau_1 \\ 0 & \text{if } t \geq \tau_1 \end{cases}$$

$$\text{and if } i > 2 \beta_i^t(\tau_1, \dots, \tau_i) =$$

$$= \begin{cases} i & \text{if } t < \alpha_1^i \\ -i k & \text{if } \alpha_1^i + \dots + \alpha_k^i \leq t < \alpha_1^i + \dots + \alpha_{k+1}^i \quad 0 < k \leq i-1 \\ 0 & \text{if } t \geq \alpha_1^i + \dots + \alpha_i^i \end{cases}$$

If we suppose to be i customers at the beginning and there are no later arrivals, α_k^i is the k th period of time since their service started, during which the number of customers in the system is constant. The same suppositions being made, β_i^t is the number of customers in the system at the time t since the service started.

$\beta_i^{i+1}(\tau_1, \dots, \tau_{i+1})$ is characterized by $\alpha_1^{i+1}, \dots, \alpha_{i+1}^{i+1}$ where

$$\alpha_k^{i+1} = \alpha_k^i \quad k \leq i-1$$

$$\alpha_i^{i+1} = \min(\alpha_i^i, \tau_{i+1})$$

$$\alpha_{i+1}^{i+1} = \max(\alpha_i^i, \tau_{i+1}) - \min(\alpha_i^i, \tau_{i+1})$$

Proposition 1.2: If $\tau_1, \dots, \tau_i$ are independent, identically distributed with the distribution e_λ random variables, then $\alpha_1^i, \dots, \alpha_i^i$ defined in Remark 1.2 are independent random variables with the distributions $e_{2\lambda}, \dots, e_{2\lambda}, e_\lambda$.

Proof: We are going to prove this proposition using induction on i . If $i = 1$, $\alpha_1^1 = \tau_1$ which has the distribution e_λ . Let this assertion be demonstrated for $k \leq i$. Hence $\alpha_1^i, \dots, \alpha_i^i$ are independent with the distributions $e_{2\lambda}, \dots, e_{2\lambda}, e_\lambda$.

$\mathcal{B}(\alpha_1^i, \dots, \alpha_i^i) \subset \mathcal{B}(\tau_1, \dots, \tau_i)$ and $\mathcal{B}(\tau_{i+1})$ and $\mathcal{B}(\tau_1, \dots, \tau_i)$ are independent. Therefore $\mathcal{B}(\tau_{i+1})$ and $\mathcal{B}(\alpha_1^i, \dots, \alpha_i^i)$ are independent.

Because $\alpha_1^i, \dots, \alpha_i^i$ are independent, using the desassociativity property it follows that $\alpha_1^i, \dots, \alpha_i^i, \tau_{i+1}$ are independent. Denote by $\gamma_1 = \alpha_i^i$ and $\gamma_2 = \tau_{i+1}$. We are going to prove that $\alpha_{i+1}^{i+1} = \min(\gamma_1, \gamma_2)$ and $\alpha_{i+1}^{i+1} = \max(\gamma_1, \gamma_2) - \min(\gamma_1, \gamma_2)$ are independent and have the distributions $e_{2\lambda}, e_\lambda$. Due to Lemma 1.1. $\gamma_1 = \min(\gamma_1, \gamma_2)$ and $\gamma_2 - \gamma_1 = \max(\gamma_1, \gamma_2) - \min(\gamma_1, \gamma_2)$ are independent and have the distributions $e_{2\lambda}, e_\lambda$ with respect to $P_{(\gamma_1 = \min(\gamma_1, \gamma_2))} = P_{(\gamma_2 > \gamma_1)}$; $\gamma_2 = \min(\gamma_1, \gamma_2)$ and $\gamma_1 - \gamma_2 = \max(\gamma_1, \gamma_2) - \min(\gamma_1, \gamma_2)$ are independent and have the distributions $e_{2\lambda}, e_\lambda$ with respect to $P_{(\gamma_2 < \gamma_1)}$. Due to Lemma 2.1. $\min(\gamma_1, \gamma_2)$ and $\max(\gamma_1, \gamma_2) - \min(\gamma_1, \gamma_2)$ are independent and have the distributions $e_{2\lambda}, e_\lambda$ with respect to P .

Therefore $\alpha_1^{i+1}, \dots, \alpha_{i+1}^{i+1}$ are independent and have the distributions $e_{2\lambda}, \dots, e_{2\lambda}, e_\lambda$. We also used the desassociativity property for $\mathcal{B}(\alpha_1^i, \dots, \alpha_i^i), \mathcal{B}(\alpha_{i+1}^{i+1}, \alpha_{i+1}^{i+1})$ which are independent.

Remark 2.2: Let $\{E, \mathcal{K}, P\}$ be a probability space and τ be a e_λ distributed random variable on it, such that $\mathcal{B}(\tau)$ and $\mathcal{B}(\beta_s^i, s \in [0, \infty))$ are independent.

Denote by $\gamma_t((\beta_s^i)_{s \in [0, \infty)}, \tau) = \beta_t^{i+1}((\beta_s^i)_{s \in [0, \infty)}, \tau)$, $\gamma = (\gamma_t)_{t \in [0, \infty)}$. We have $\beta_t^{i+1}(\gamma_1, \dots, \tau_{i+1}) = \gamma_t((\beta_s^i(\tau_1, \dots, \tau_i))_{s \in [0, \infty)}, \tau_{i+1})$ and $(\beta_s^i)_{s \in [0, \infty)}$ depends on $\alpha_1^i, \dots, \alpha_i^i$; therefore $\gamma(\beta_s^i(\tau_1, \dots, \tau_i)_{s \in [0, \infty)}, \tau_{i+1})$ depends on $\alpha_1^{i+1}, \dots, \alpha_{i+1}^{i+1}$.

Remark 3.2: a. $(\beta_t^i(\tau_1, \dots, \tau_i))_{t \in [0, \infty)}$ defined in Remark 1.2. is a minimal process, corresponding to the minimal semigroup $(Q_t)_{t > 0}$ given by $Q = (q_{ij})_{i, j \in I}$ and $\Lambda = (\lambda_i)_{i \in I}$, $I = \{0, 1, \dots\}$, where $q_{ii-1} = 1$, $q_{00} = 1$, $q_{ij} = 0$ otherwise and $\lambda_i = 2\lambda$ if $i > 1$, $\lambda_1 = \lambda$. Furthermore it is a death process.

b. $(\beta_t^i(\tau_1, \dots, \tau_i))_{t \in [0, \infty)}$ is a (Q_t) process with right continuous sample paths.

Proof: This assertion follows from the construction of the minimal process in [4] and Lemma 3.1.

Remark 4.2: We have the equivalence between Remark 3.2. b. and the following one: $(\beta_t^i(\tau_1, \dots, \tau_i))_{t \in [0, \infty)}$ is a right continuous sample paths process and $M(\chi_{\{j\}} | \beta_{t+s}^i | \mathcal{B}(\beta_u^i | u \leq s)) = Q_t(\beta_s^i, \{j\})$

Proof: See [4].

Remark 5.2 Let us use the strong Markov property for $(\beta_t^i)_{t \in [0, \infty)}$ defined in Remark 1.2

Let σ be a positive random variable, such that σ and $\mathcal{B}(\beta_t^i, t \in [0, \infty))$ are independent. Lemma 5.1 for $\mathcal{G} = \mathcal{B}(\sigma)$ applied to Remark 4.2 leads us to: $M(\chi_{\{j\}} \circ \beta_{t+s}^i | \mathcal{B}(\beta_u^i, \sigma | u \leq s)) = Q_t(\beta_s^i, \{j\})$.

Denote by $\mathcal{F} = \mathcal{B}(\beta_u, \sigma | u \leq s)$. Let ξ be a $(\mathcal{F}_s)_{s \in [0, \infty)}$ stopping time. Then $M(\chi_{\{j\}} \circ \beta_{t+\xi}^i | \mathcal{F}_\xi) = Q_t(\beta_\xi^i, \{j\})$. (the strong Markov property [1]).

Proposition 2.2: $(\beta_{t+\xi}^i)_{t \geq 0}$ is a right continuous sample paths (Q_t) process, $(Q_t)_{t \geq 0}$ defined in Remark 3.2. Furthermore it is a death process.

Proof: If we apply Remark 5.2. for $\xi + s, s > 0$, we obtain :

$M(\chi_{\{j\}} \circ \beta_{t+\xi+s}^i | \mathcal{F}_{\xi+s}) = Q_t(\beta_{\xi+s}^i, \{j\})$ $\beta_{\xi+u}^i$ being $\mathcal{F}_{\xi+u}$ measurable and $\mathcal{F}_{\xi+u} \subset \mathcal{F}_{\xi+s}$, we have $\mathcal{B}(\beta_{\xi+u}^i | u \leq s) \subset \mathcal{F}_{\xi+s}$; therefore
 $M(M(\chi_{\{j\}} \circ \beta_{\xi+t+s}^i | \mathcal{F}_{\xi+s}) | \mathcal{B}(\beta_{\xi+u}^i | u \leq s)) = M(Q_t(\beta_{\xi+s}^i, \{j\}) | \mathcal{B}(\beta_{\xi+u}^i | u \leq s))$
Hence $M(\chi_{\{j\}} \circ \beta_{t+\xi+s}^i | \mathcal{B}(\beta_{\xi+u}^i | u \leq s)) = Q_t(\beta_{\xi+s}^i, \{j\})$.

Remark 6.2 Denote by β_t the number of customers in the system at the time t since the service started, supposing there are no later arrivals — a random variable on $\{E, \mathcal{K}, P\}$; by $\sigma_1, \dots, \sigma_n, \dots$ the interarrivals times after the service started (σ_1 is the period of time from the beginning of the service until the first arrival) — independent, F distributed random variables;

by $\tau_1, \dots, \tau_n, \dots$ the service times of the customers who are going to arrive after the service started — independent, e_λ ($\lambda > 0$) distributed random variables;

by y_k the number of customers in the system at the time $\sigma_1 + \dots + \sigma_k$ ($k \geq 1$); $y_0 = \beta_0$;

by β_t^{*k} the number of customers in the system at the time $t + \sigma_1 + \dots + \sigma_k$, supposing that the arrivals $k+1, k+2, \dots$ did not take place ($t > 0$). $(\beta_t^{*k})_{t \in [0, \infty)}$ is the service process of the y_k customers.

Remark that :

$$\beta_t^{*k} = \beta_t^{*k}((\beta_s)_{s \in [0, \infty)}, \sigma_1, \dots, \sigma_n, \dots, \tau_1, \dots, \tau_n, \dots)$$

$$y_k = y_k((\beta_s)_{s \in [0, \infty)}, \sigma_1, \dots, \sigma_n, \dots, \tau_1, \dots, \tau_n, \dots) =$$

$$= \beta_0^{*k}((\beta_s)_{s \in [0, \infty)}, \sigma_1, \dots, \sigma_n, \dots, \tau_1, \dots, \tau_n, \dots)$$

$$\beta_t^{*0} = \beta_t$$

$$\beta_t^{*k+1}((\beta_s)_{s \in [0, \infty)}, \sigma_1, \dots, \sigma_n, \dots, \tau_1, \dots, \tau_n, \dots) =$$

$$= \beta_t^{*k}(\gamma(\beta_{s+\sigma_1})_{s \in [0, \infty)}, \tau_1, \sigma_n, \dots, \sigma_n, \dots, \tau_2, \dots, \tau_n, \dots)$$

$$\text{where } \beta_t(\omega) = \begin{cases} i & \text{if } t < \alpha_1(\omega) \\ i-1 & \text{if } \alpha_1(\omega) \leq t < \alpha_1(\omega) + \alpha_2(\omega) \\ \vdots & \\ 1 & \text{if } \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) \leq t < \alpha_1(\omega) + \dots + \alpha_i(\omega) \\ 0 & \text{if } t \geq \alpha_1(\omega) + \dots + \alpha_i(\omega) \end{cases}$$

$$(i = i(\omega), \omega \in E)$$

α_k is the k th period of time since the service started during which the number of customers is constant, supposing there are no later arrivals.

we have defined the process $\gamma = (\gamma_t (\beta_s)_{s \in [0, \infty)} \tau)_{t \in [0, \infty)}$ for τ e_λ distributed random variable such that $\tau, (\beta_t)_{t \in [0, \infty)}, \sigma_1, \dots, \sigma_n, \dots$ are independent:

$$\begin{aligned} \text{if } i=0 \quad \gamma_t((\beta_s)_{s \in [0, \infty)}(\omega), \tau(\omega)) &= \begin{cases} 1 & \text{if } t < \tau(\omega) \\ 0 & \text{if } t > \tau(\omega) \end{cases} \\ \text{if } i=1 \quad \gamma_t((\beta_s)_{s \in [0, \infty)}(\omega), \tau(\omega)) &= \begin{cases} 2 & \text{if } t < \min \tau(\alpha_1(\omega), \tau(\omega)) \\ 1 & \text{if } \min(\alpha_1(\omega), \tau(\omega)) \leq t < \max(\alpha_1(\omega), \tau(\omega)) \\ 0 & \text{if } t > \max(\alpha_1(\omega), \tau(\omega)) \end{cases} \\ \text{if } i \geq 2 \quad \gamma_t((\beta_s)_{s \in [0, \infty)}(\omega), \tau(\omega)) &= \begin{cases} \beta_t(\omega) + 1 & \text{if } t < \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) \\ 2 & \text{if } \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) \leq t < \alpha_1(\omega) + \dots \\ & \dots + \alpha_{i-1}(\omega) + \min(\alpha_i(\omega), \tau(\omega)) \\ 1 & \text{if } \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) + \min(\alpha_i, \tau(\omega)) < \\ & \leq t < \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) + \max(\alpha_i(\omega), \tau(\omega)) \\ 0 & \text{if } t > \alpha_1(\omega) + \dots + \alpha_{i-1}(\omega) + \\ & + \max(\alpha_i(\omega), \tau(\omega)) \end{cases} \end{aligned}$$

Proposition 3.2: Let $(\beta_t)_{t \in [0, \infty)}$ be the (Q_t) process defined in Remark 6.2. Then $\gamma((\beta_s)_{s \in [0, \infty)}, \tau)$ is a (Q_t) process. Moreover, they are death processes. (In fact they are minimal processes corresponding to $Q = (q_{ij})_{i,j \in I}, q_{i,i-1} = 1, q_{00} = 1, q_{ij} = 0$ otherwise and $\Lambda = (\lambda_i)_{i \in I}, \lambda_i = 2\lambda$ if $i > 1, \lambda_1 = \lambda, I = \{0, 1, 2, \dots\}$)

Proof: $(\beta_t)_{t \in [0, \infty)}$ is a (Q_t) process with respect to P . Hence it is a (Q_t) process with respect to $P_{(\beta_0=i)}$. (Lemma 4.1)

It follows from Remark 1.1. that τ is e_λ distributed and τ and $(\beta_t)_{t \in [0, \infty)}$ — similar to τ_{t+1} (e_λ distributed) and $(\beta_t)_{t \in [0, \infty)}$ defined in Remark 1.2. — are independent with respect to $P_{(\beta_0=i)}$. Hence the distribution of $\gamma_t((\beta_s)_{s \in [0, \infty)}, \tau)$ is the same with the distribution of β_t^{i+1} (With respect to $P_{(\beta_0=i)}$)

According to Remark 1.2. and Remark 3.2. $(\gamma_t (\beta_s)_{s \in [0, \infty)} \tau)_{t \in [0, \infty)}$ is a (Q_t) process with respect to $P_i = P_{(\beta_0=i)}$. We have $P = \sum_{i \in I} P_{(\beta_0=i)}$

P_i and $M_{P_i}(\chi_{ij} \circ \gamma_{t+s} | \mathcal{B}(\gamma_u | u \leq s)) = Q_t(\gamma_s, \{j\})$; using Remark 4.2. it follows that $\gamma((\beta_s)_{s \in [0, \infty)}, \tau)$ is a (Q_t) process with respect to P .

Proposition 4.2: Let $\{E^i, P^i\}$ be a probability space for every $i \in I$ ($I = \{0, 1, 2, \dots\}$) such that $\beta_0 = i$ on $E^i, (\beta_t)_{t \in [0, \infty)}$ being the process defined in Remark 6.2. Denote by y_k^i the number of customers at the time $\sigma_1 + \dots + \sigma_k$, supposing there were i customers at the beginning of the service. Then $(y_k^i)_{k=0,1,\dots}$ is a homogenous Markov chain corresponding to $Q = (q_{ij})_{i,j \in I}, q_{ij} P^i(y_1^i = j)$.

Proof. According to Proposition 1.1. it is sufficient to prove that the distribution of $(y_1^i, \dots, y_n^i)$ with respect to $P_{(\beta_1^i=j)}$ is the same with the distribution of $(y_0^j, y_1^j, \dots, y_n^j, \dots)$ with respect to P^j .

We have: $y_0^j = i$ P^i a.e. ($y_0^i = \beta_0^{*0} = \beta_0 = i$ on E^i); ($y_1 = j$) = ($\beta_0^{*1} = j$);

$$\begin{aligned} & \beta_0^{*1}((\beta_s)_{s \in [0, \infty)} \sigma_1, \dots, \sigma_n, \dots, \tau_1, \dots, \tau_n, \dots) = \\ & = \beta_0^{*0}(\gamma((\beta_{\sigma_1+s})_{s \in [0, \infty)} \tau_1), \sigma_2, \dots, \sigma_n, \dots, \tau_2, \dots, \tau_n, \dots) = \\ & = \gamma_0((\beta_{\sigma_1+s})_{s \in [0, \infty)}, \tau_1) = \beta_{\sigma_1} + 1; \end{aligned}$$

($\beta_0^{*1} = j$) = ($\beta_{\sigma_1} = j - 1$). Hence $P_{(y_1^i=j)} = P_{(\beta_{\sigma_1}^i=j-1)}$. (We have denoted by σ_1^i, σ_1 for the process which starts from i and $(\beta_t^i)_{t \in [0, \infty)}$ is the process defined in Remark 1.2.)

It follows from Proposition 2.2, Remark 5.2 and $(\sigma_1^i \leq s) \in \mathcal{F}_s = \mathcal{B}(\beta_u^i, \sigma_1^i \mid u \leq s)$ that $(\beta_{t+\sigma_1^i}^i)_{t \in [0, \infty)}$ is a (Q_t) process with respect to P^i , $(Q_t)_{t \geq 0}$ given by Remark 3.2. According to Lemma 4.1. $(\beta_{t+\sigma_1^i}^i)_{t \in [0, \infty)}$ is

a (Q_t) (death) process with respect to $P_{(\beta_{\sigma_1^i}^i=j-1)}$ which starts from $j - 1$.

The fact that $(\beta_{\sigma_1^i}^i = j - 1) \in \mathcal{B}(\beta_{s+\sigma_1^i}^i \mid s \in [0, \infty))$ enables us to apply Remark 1.1 and so we have:

- a. $(\beta_{s+\sigma_1^i}^i)_{s \in [0, \infty)}, \tau_1^i, \dots, \tau_n^i, \dots$ are independent and τ_k^i has the distribution e_λ for every $k \geq 1$.
- b. $\sigma_2^i, \dots, \sigma_n^i$ are independent and F distributed,
- c. this set of random variables are altogether independent, with respect to $P_{(\beta_{\sigma_1^i}^i=j-1)}$.

According to Proposition 3.2. $\gamma((\beta_{s+\sigma_1^i}^i)_{s \in [0, \infty)}, \tau_1^i)$ is a (Q_t) (death) process with respect to $P_{(\beta_{\sigma_1^i}^i=j-1)}$ which starts from j of the same type with $(\beta_{t+\sigma_1^i}^i)_{t \in [0, \infty)}$ and we have:

- a'. $\gamma((\beta_{s+\sigma_1^i}^i)_{s \in [0, \infty)}, \tau_1^i), \tau_2^i, \dots, \tau_n^i, \dots$ are independent and τ_k^i has the distribution e_λ for every $k \geq 2$.
- b'. $\sigma_2^i, \dots, \sigma_n^i, \dots$ are independent and F distributed,
- c'. this set of random variables are altogether independent, with respect to $P_{(\beta_{\sigma_1^i}^i=j-1)}$.

Therefore the distributions of

$(\beta_t^{*k+1}((\beta_s^i)_{s \in [0, \infty)} \sigma_1^i, \dots, \sigma_n^i, \dots, \tau_1^i, \dots, \tau_n^i, \dots))$ $k = 0, 1, \dots$ with respect to $P_{(y_1^i=j)}$ of $(\beta_t^{*k}(\gamma((\beta_{s+\sigma_1^i}^i)_{s \in [0, \infty)}, \sigma_1^i), \sigma_2^i, \dots, \sigma_n^i, \dots, \tau_2^i, \dots, \tau_n^i, \dots), \dots))$, $k = 0, 1, \dots$ with respect to $P_{(y_1^i=j)}$, and of $(\beta_t^{*k}((\beta_s^j)_{s \in [0, \infty)}, \sigma_1^j, \dots, \sigma_n^j, \dots, \tau_1^j, \dots, \tau_n^j, \dots))$, $k = 0, 1, \dots$ with respect to P^j

are the same. That means that the distribution of $(\beta_i^{*1}, \dots, \beta_i^{*n}, \dots)$ with respect to $P_{(y_1^i=j)}^i$ is the same with the distribution of $(\beta_i^{*0}, \dots, \beta_i^{*n}, \dots)$ with respect to P^j . If we take $t = 0$ and we apply Proposition 1.1 the proposition is proved.

§ 3. The Queueing System G/M/n

Remark 1.3: In the case of the queueing system G/M/3 if Proposition 1.2 remains true, then all the arguments used to prove the results in §2. are valid. In this new situation Proposition 1.2 becomes: „if $\tau_1, \dots, \tau_i$ are independent, e_λ distributed random variables, then $\alpha_1^i, \dots, \alpha_i^i$ are independent random variables with the distributions $e_{3\lambda}, \dots, e_{3\lambda}, e_{2\lambda}, e_\lambda$. (Remark that $\tau_1, \dots, \tau_i$ and $\alpha_1^i, \dots, \alpha_i^i$ have the same meaning as in Remark 1.2, but the relation between them are different). The process $(\beta_i^t)_{t \in [0, \infty)}$ defined in Remark 1.2. becomes the minimal process corresponding to the minimal semigroup $(Q_t)_{t \geq 0}$ given by $Q = (q_{ij})_{i,j \in I}$ and $\Lambda = (\lambda_i)_{i \in I}$, $I = \{0, 1, \dots\}$ where $q_{i,i-1} = 1$ if $i > 1$, $q_{00} = 1$, $q_{ij} = 0$ otherwise, $\lambda_i = 3\lambda$ if $i > 2$, $\lambda_2 = 2\lambda$, $\lambda_1 = \lambda$.

In order to prove Proposition 1.2 for the queueing system G/M/3 it will be sufficient to verify the i th induction step, that is: if $\alpha_{i-1}^i, \alpha_i^i, \tau_{i+1}$ are independent with the distributions $e_{2\lambda}, e_\lambda, e_\lambda$ then $\alpha_{i-1}^{i+1}, \alpha_i^{i+1}, \alpha_{i+1}^{i+1}$ are independent with the distributions $e_{3\lambda}, e_{2\lambda}, e_\lambda$. (If $i = 1$ Proposition 1.2 for G/M/3 coincides with that for G/M/2. So it is possible to consider $i \geq 3$).

Denote by: $\alpha_1 = \alpha_{i-1}^i$, $\alpha_2 = \alpha_i^i$, $\tau = \tau_{i+1}$, $\gamma_1 = \alpha_{i-1}^{i+1}$, $\gamma_2 = \alpha_i^{i+1}$, $\gamma_3 = \alpha_{i+1}^{i+1}$

We have: $\gamma_1 = \min(\alpha_1, \tau)$;

$$(1) \quad \gamma_2 = \begin{cases} \min(\alpha_2, \tau - \alpha_1) & \text{if } \alpha_1 \leq \tau; \\ \alpha_1 - \tau & \text{if } \alpha_1 > \tau; \end{cases}$$

$$\begin{cases} \alpha_2 & \text{if } \tau < \alpha_1 \\ \alpha_1 + \alpha_2 - \tau & \text{if } \alpha_1 \leq \tau < \alpha_1 + \alpha_2 \\ \tau - (\alpha_1 + \alpha_2) & \text{if } \tau > \alpha_1 + \alpha_2 \end{cases}$$

Definition 1.3: Let $\tau_1, \dots, \tau_{n+1}$ be independent, e_λ distributed random variables. We define $\beta_2: [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$

$$\beta_2(\tau_1, \tau_2) = (\min(\tau_1, \tau_2), \max(\tau_1, \tau_2) - \min(\tau_1, \tau_2));$$

$$\beta_{n+1}: [0, \infty) \times \dots \times [0, \infty) \rightarrow [0, \infty) \times \dots \times [0, \infty)$$

$n+1 \text{ times} \qquad \qquad \qquad n+1 \text{ times}$

$$\beta_{n+1}(\tau_1, \dots, \tau_{n+1}) = (\min(\tau_1, \dots, \tau_{n+1}), \beta_n(\tau_1 - \min(\tau_1, \dots, \tau_{n+1}), \dots, \tau_{n+1} - \min(\tau_1, \dots, \tau_{n+1}))) \quad n \geq 2$$

„/” marks the place where $\tau_i - \min(\tau_1, \dots, \tau_{n+1})$ is zero. We omit this term.

Lemma 1.3: Let β_n be the random vector defined above. Then we have: $(e_\lambda \otimes \dots \otimes e_\lambda) \circ \beta_n^{-1} = e_{n\lambda} \otimes e_{(n-1)\lambda} \otimes \dots \otimes e_\lambda$.

Proof: We will carry out the proof by induction on n .

For $n = 2$ the relation to prove is : $(e_\lambda \otimes e_\lambda) \circ \beta_2^{-1} = e_{2\lambda} \otimes e_\lambda$.

Due to Lemma 1.1 $\min(\tau_1, \tau_2)$ and $\max(\tau_1, \tau_2) - \min(\tau_1, \tau_2)$ are independent with the distributions $e_{2\lambda}, e_\lambda$ with respect to $P_{(\tau_1 < \tau_2)}$. Similarly they are independent with the distributions $e_{2\lambda}, e_\lambda$ with respect to $P_{(\tau_2 < \tau_1)}$ and by Lemma 2.1, with respect to P .

Let this assertion be demonstrated for $k \leq n$,

According to Lemma 1.1 for $P_{(\tau_i = \min(\tau_1, \dots, \tau_{n-1}))}$ and Lemma 2.1 we have : $\min(\tau_1, \dots, \tau_{n+1}), \tau_1 - \min(\tau_1, \dots, \tau_{n+1}), \dots, \tau_{n+1} - \min(\tau_1, \dots, \tau_{n+1})$ are independent with the distributions $e_{(n+1)\lambda}, e_\lambda, \dots, e_\lambda$, with respect to P . Denote by $f, z = (pr_1, \beta_n \circ pr_2)$ the mappings given by :

$$\begin{aligned} (\tau_1, \dots, \tau_{n+1}) &\xrightarrow{f} (\min \tau_1, \dots, \tau_{n+1}), \tau_1 - \min(\tau_1, \dots, \tau_{n+1}), \dots, \tau_{n+1} - \\ &\quad - \min(\tau_1, \dots, \tau_{n+1}) \xrightarrow{z} (\min(\tau_1, \dots, \tau_{n+1}), \\ &\quad \beta_n(\tau_1 - \min(\tau_1, \dots, \tau_{n+1}), \dots, \tau_{n+1} - \min(\tau_1, \dots, \tau_{n+1}))) \end{aligned}$$

where pr_1 and pr_2 are the natural projections on the first coordinate and on the other n , respectively. Hence we have $z \circ f = \beta_{n+1}$. It follows that $(e_\lambda \otimes \dots \otimes e_\lambda) \circ f^{-1} = e_{(n+1)\lambda} \otimes e_\lambda \otimes \dots \otimes e_\lambda$. Therefore and due to the induction hypothesis $(e_\lambda \otimes \dots \otimes e_\lambda) \circ \beta_{n+1}^{-1} = (e_\lambda \otimes \dots \otimes e_\lambda) \circ f^{-1} \circ z^{-1} = ((e_{(n+1)}) \otimes (e_\lambda \otimes \dots \otimes e_\lambda)) \circ (pr_1, \beta_n \circ pr_2)^{-1} = (e_{(n+1)\lambda}) \otimes (e_{n\lambda} \otimes \dots \otimes e_\lambda)$

Remark 2.3: Let $\{E^\circ, \mathcal{K}^\circ, P^\circ\}$ be a probability space and τ_1, τ_2, τ_3 be independent, e_λ distributes random variables on it.

According to Lemma 1.3. $\beta_3(\tau_1, \tau_2, \tau_3)$ is a random vector with independent components, their distributions being $e_{3\lambda}, e_{2\lambda}, e_\lambda$.

Let $\alpha_1^0, \alpha_2^0, \tau_3$ be independent random variables on $\{E^\circ, \mathcal{K}^\circ, P^\circ\}$ having the distributions $e_{2\lambda}, e_\lambda, e_\lambda$ such that $(\alpha_1^0, \alpha_2^0) = \beta_2(\tau_1, \tau_2)$.

Then $(\gamma_1^0, \gamma_2^0, \gamma_3^0) = \beta_3(\tau_1, \tau_2, \tau_3)$. $(\gamma_1^0, \gamma_2^0, \gamma_3^0)$ defined in Remark 1.3. (I) for $\tau = \tau_3$

An easy calculation considering all the possible choices for τ_1, τ_2, τ_3 leads us to this equality.

Hence on the probability space on which τ_1, τ_2, τ_3 are independent and e_λ distributed we have :

- $\alpha_1^0, \alpha_2^0, \tau_3$ are independent with the distributions $e_{2\lambda}, e_\lambda, e_\lambda$;
- $\gamma_1^0, \gamma_2^0, \gamma_3^0$ are independent with the distributions $e_{3\lambda}, e_{2\lambda}, e_\lambda$.

If we consider another probability space $\{E', \mathcal{K}', P'\}$ on which $\alpha'_1, \alpha'_2, \tau'_3$ are independent random variables with the distributions $e_{2\lambda}, e_\lambda, e_\lambda$ and we define $\gamma'_1, \gamma'_2, \gamma'_3$ as verifying the relations (1) from Remark 1.3 we find that $\gamma'_1, \gamma'_2, \gamma'_3$ are independent and have the distributions $e_{3\lambda}, e_{2\lambda}, e_\lambda$.

Hence Proposition 1.2 is valid for the queueing system $G/M/3$ and $(y_k^i)_{k=0,1,\dots}$ defined in Remark 6.2 is a homogenous Markov chain.

Remark 3.3 If we consider now the queueing system $G/M/n$ in order to demonstrate that $(y_k^i)_{k=0,1,\dots}$ (defined in Remark 6.2) is a Markov chain

we are going to use induction on n (the number of servers). As we have shown in Remark 1.3 it would be sufficient to prove Proposition 1.2 from § 2. in the n servers case.

If $n = 1$ Proposition 1.2 is obviously true. In this case $(\beta_i^t)_{t \in [0, \infty)}$ defined in Remark 1.2 becomes the minimal process corresponding to the minimal semigroup $(Q_t)_{t \geq 0}$ given by

$$Q = (q_{ij})_{i,j \in I} \text{ and } \Lambda = (\lambda_i)_{i \in I} \quad I = \{0, 1, \dots\}, \quad q_{i,i-1} = 1 \text{ if } i > 1,$$

$q_{00} = 1, q_{ij} = 0$ otherwise, $\lambda_i = \lambda$ for every $i \in I$.

Let the statement of Proposition 1.2 be fulfilled for all $k \leq n$. In the case of $n + 1$ servers Proposition 1.2 becomes :

"If $\tau_1, \dots, \tau_i$ are independent, e_λ distributed random variables then $\alpha_1^i, \dots, \alpha_i^i$ are independent random variables with the distributions $e_{(n+1)\lambda}, \dots, e_{(n+1)\lambda}, e_{n\lambda}, \dots, e_\lambda$." ($\tau_1, \dots, \tau_i$ and $\alpha_1^i, \dots, \alpha_i^i$ have the same meaning as in Remark 1.2, but the relations between them are different). If Proposition 1.2 holds for $G/M/n + 1$ $(\beta_i^t)_{t \in [0, \infty)}$ (Remark 1.2) will become the minimal process corresponding to the minimal semigroup $(Q_t)_{t \geq 0}$ given by $Q = (q_{ij})_{i,j \in I}$ and $\Lambda = (\lambda_i)_{i \in I}$, $I = \{0, 1, \dots\}$, $q_{i,i-1} = 1$ if $i > 1$, $q_{00} = 1$, $q_{ij} = 0$, otherwise, $\lambda_i = (n + 1) \lambda$ if $i > n$, $\lambda_n = \lambda, \dots, \lambda_1 = \lambda$.

We can reduce the proof of Proposition 1.2 for $G/M/n + 1$ to the analyse of its i th induction step, that is : "if $\alpha_{i-n+1}^i, \dots, \alpha_i^i, \tau_{i+1}$ are independent and with the distributions $e_{n\lambda}, \dots, e_{2\lambda}, e_\lambda, e_\lambda$ then $\alpha_{i-n+1}^{i+1}, \alpha_{i-n+2}^{i+1}, \dots, \alpha_{i+1}^{i+1}$ are independent with the distributions $e_{(n+1)\lambda}, e_{n\lambda}, \dots, e_\lambda$." (If $i \leq n$ Proposition 1.2 for $G/M/n + 1$ coincides with that for $G/M/n$ which is supposed to be true by the induction hypothesis. So we may consider $i > n$.)

Denote by : $\alpha_1 = \alpha_{i-n+1}^i, \dots, \alpha_n = \alpha_i^i, \tau = \tau_{i+1}$,

$$\gamma_1 = \alpha_{i-n+1}^{i+1}, \dots, \gamma_{n+1} = \alpha_{i+1}^{i+1}.$$

We have : $\gamma_1 = \min (\alpha_1, \tau)$

$$(2) \quad \begin{aligned} \gamma_2 &= \begin{cases} \alpha_1 - \tau & \text{if } \tau < \alpha_1 \\ \min (\tau - \alpha_1, \alpha_2) & \text{if } \tau > \alpha_1 \end{cases} \\ \gamma_3 &= \begin{cases} \alpha_2 & \text{if } \tau < \alpha_1 \\ \alpha_1 + \alpha_2 - \tau & \text{if } \alpha_1 \leq \tau < \alpha_1 + \alpha_2 \\ \tau - (\alpha_1 + \alpha_2) & \text{if } \alpha_1 + \alpha_2 \leq \tau < \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_3 & \text{if } \tau > \alpha_1 + \alpha_2 + \alpha_3 \end{cases} \\ \vdots & \\ \gamma_i &= \begin{cases} \alpha_{i-1} & \text{if } \tau < \alpha_1 + \dots + \alpha_{i-2} \\ \alpha_1 + \dots + \alpha_{i-1} - \tau & \text{if } \alpha_1 + \dots + \alpha_{i-2} \leq \tau < \alpha_1 + \dots + \alpha_{i-1} \\ \tau - (\alpha_1 + \dots + \alpha_{i-1}) & \text{if } \alpha_1 + \dots + \alpha_{i-1} \leq \tau < \alpha_1 + \dots + \alpha_i \\ \alpha_i & \text{if } \tau > \alpha_1 + \dots + \alpha_i \end{cases} \\ \vdots & \end{aligned}$$

$$(2) \quad \gamma_{n+1} = \begin{cases} \alpha_n & \text{if } \tau < \alpha_1 + \dots + \alpha_{n-1} \\ \alpha_1 + \dots + \alpha_n - \tau & \text{if } \alpha_1 + \dots + \alpha_{n-1} \leq \tau < \alpha_1 + \dots + \alpha_n \\ \tau - (\alpha_1 + \dots + \alpha_n) & \text{if } \tau \geq \alpha_1 + \dots + \alpha_n \end{cases}$$

Let $\{E^\circ, \mathcal{H}^\circ, P^\circ\}$ be a probability space and $\tau_1, \dots, \tau_{n+1}$ be independent, e_λ distributed random variables on it. Then $\beta_{n+1}(\tau_1, \dots, \tau_{n+1})$ is a random vector with independent components their distributions being $e_{(n+1)\lambda}, \dots, e_\lambda$.

Due to Remark 2.3 and Lemma 1.3 it is sufficient to prove that if $\alpha_1^0, \dots, \alpha_n^0, \tau_{n+1}$ are independent random variables on $\{E^\circ, \mathcal{H}^\circ, P^\circ\}$ with the distributions $e_{n\lambda}, \dots, e_{2\lambda}, e_\lambda, e_\lambda$ such that $(\alpha_1^0, \dots, \alpha_n^0) = \beta_n(\tau_1, \dots, \tau_n)$ then $(\gamma_1^0, \dots, \gamma_{n+1}^0) = \beta_{n+1}(\tau_1, \dots, \tau_{n+1})$. $(\gamma_1^0, \dots, \gamma_{n+1}^0)$ given by (2) for $\alpha_1^0, \dots, \alpha_n^0$ and $\tau = \tau_{n+1}$.

In order to prove this assertion let us define on $\{E^\circ, \mathcal{H}^\circ, P^\circ\}$ the following random variables:

$\tau_1, \dots, \tau_{n+1}$ the service times of the first $n+1$ customers (we suppose to be i customers in the system at the beginning of the service, $i \geq n+1$), $\gamma_1^0, \dots, \gamma_{n+1}^0$ the periods of time during which the service process of the first $n+1$ customers is constant (the services $n+2, \dots, i$ are not considered and it is supposed to be no later arrivals).

$\alpha_1^0, \dots, \alpha_n^0$ the periods of time during which the service process of the first n customers is constant (the services $n+1, \dots, i$ and later arrivals are not considered).

Obviously we have: $(\alpha_1^0, \dots, \alpha_n^0) = \beta_n(\tau_1, \dots, \tau_n)$, $(\gamma_1^0, \dots, \gamma_{n+1}^0) = \beta_{n+1}(\tau_1, \dots, \tau_{n+1})$ and $\gamma_1^0, \dots, \gamma_{n+1}^0, \alpha_1^0, \dots, \alpha_n^0, \tau_{n+1}$ verify the relations (2).

(Remark that we have implicitly used the same type of construction in Remark 2.3).

The required result is then proved.

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Mathematical Faculty
University of Bucharest

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SUR LA STRUCTURE DES ALGÈBRES LOCALES NOETHÉRIENNES FORMELLEMENT LISSES.

N. RADU

Tous les anneaux considérés sont supposés commutatifs, unitaires et les morphismes d'anneaux sont supposés unitaires. On dit qu'un morphisme d'anneaux $u : A \rightarrow B$ est un morphisme d'anneaux locaux si les anneaux A et B sont locaux et l'image par u de l'idéal maximal de A est contenu dans l'idéal maximal de B . Le théorème suivant sur la structure des algèbres formellement lisses a été démontré pour la première fois par A. Grothendieck dans EGA, O_{IV} , 19,7.1.

Théorème: Soient $u : A \rightarrow B$ un morphisme d'anneaux locaux noethériens et k le corps résiduel de A . Alors les deux conditions suivantes sont équivalentes:

a) *B est une A -algèbre formellement lisse, pour la topologie N -adique, où N est l'idéal maximal de B .*

b) *u est plat et $B \otimes_A k$ est une k -algèbre formellement lisse, pour la topologie $N \otimes_A k$ -adique.*

Dans EGA, O_{IV} , 19,7.1 l'implication a) $\Rightarrow$ b) est démontrée en utilisant la construction d'une A -algèbre noethérienne et plate B' telle que $B' \otimes_A k$ soit k -isomorphe à $B \otimes_A k$. D'où on déduit que B est une A -algèbre isomorphe à B , donc B est une A -algèbre plate.

Pour démontrer l'implication b) $\Rightarrow$ a), dans EGA, O_{IV} , 19,7.1., on applique un critère de lissité formelle qui utilise l'annulation d'un certain groupe d'extensions des A -algèbres. Donc pour la démonstration du théorème sont nécessaires des notions et résultats peu liés avec d'autres résultats d'algèbre commutative. Une autre démonstration qui utilise la cohomologie des algèbres commutatives est donnée dans [1], XVI. Remarquons aussi qu'il existe une connexion entre ce théorème et la structure des anneaux locaux noethériens complets.

Dans les livres d'algèbre commutative récemment parus, comme, par exemple, [4] et [5] le théorème n'est pas démontré. Dans le premier on le donne sans démonstration, tandis que dans le deuxième on démontre seulement l'implication b) $\Rightarrow$ a) et cela dans le cas particulier d'une A -algèbre de Cohen.

Dans cette note nous donnons une nouvelle démonstration du théorème, démonstration qui contient aussi une extension de l'implication b) $\Rightarrow$ a). Pour démontrer l'implication b) $\Rightarrow$ a) nous appliquerons le critère jacobien de lissité formelle (voir EGA, O_{IV} , 22.6.1; IL, XII, 2.13 et LA, 3.6), pour lequel dans IL et LA est donnée une démonstration élémentaire qui n'utilise pas le groupe Exalcom_{top}. Sont utilisés aussi la proposition O_{IV} , 19. 1. 9 de EGA (voir aussi IL, XI, 2.2 et LA, 3.3) et le lemme suivant :

Lemme: Soient C un anneau, M et I des idéaux de B tels que M soit de type fini et pour tout idéal $J \subseteq I$ de C et $x \in C$ la suite

$$J : Cx \subseteq J : Cx^2 \subseteq \dots$$

est stationnaire. Alors, pour tout nombre entier $n > 0$, il existe un nombre entier $m > 0$ tel que $M^m \cap I \subseteq M^n I$.

Démonstration. Puisque M est un idéal de type fini, il suffit de prouver que pour tout nombre entier $n > 0$ il existe un idéal J de C tel que la racine $r(J)$ de J contient M et $J \cap I \subseteq M^n I$. Pour démontrer l'existence d'un tel idéal J il suffit de prouver que l'ensemble $\mathcal{L}$ des idéaux L de C tels que $L \subseteq M^n I$ et $L \cap I \subseteq M^n I$ n'est pas vide et alors du lemme de Zorn on déduit qu'il existe dans $\mathcal{L}$ un élément maximal J . Il suffit de prouver que $r(J) \supseteq M$. En effet, supposons que $r(J) \not\supseteq M$. Alors il existe $x \in M$ tel que $x \notin r(J)$, donc $x^s \notin r(J)$, pour tout entier $s > 0$. Par hypothèse il existe un entier $i \geq 0$ tel que $J : Cx^i = J : Cx^{i+j}$, pour tout $j > 0$. On obtient $J = (J + Cx^i) \cap (J : Cx^i)$, donc $(J + Cx^i) \cap I \subseteq M I^n$, ce qui est en contradiction avec le caractère maximal de J dans $\mathcal{L}$.

Nous démontrerons l'implication $b) \Rightarrow a)$ sous la forme suivante :

Proposition: Soient $u : A \rightarrow B$ un morphisme d'anneaux, M un idéal de A et N un idéal de type fini de B tel que $MB \subseteq N$.

Supposons que B/MB est une A/M -algèbre formellement lisse pour la topologie N -adique, $\text{Tor}_1^A(B, A/M) = 0$ et pour tout idéal I de B et $x \in B$ la suite croissante d'idéaux de B

$$I : Bx \subseteq I : Bx^2 \subseteq \dots \subseteq I : Bx^n \subseteq \dots$$

est stationnaire. Alors B est une A -algèbre formellement lisse pour la topologie N -adique.

Démonstration. Soient C une A -algèbre de polynômes telle qu'il existe un morphisme surjectif $v : C \rightarrow B$ et $I = \ker v$.

Soient $x_1, \dots, x_n \in C$ tels que $v(x_1), \dots, v(x_n)$ soit un système de générateurs de l'idéal N de l'anneau B et $N' = \sum_{i=1}^n Cx_i$. Alors la topologie N -adique de B est la topologie quotient de la topologie N' -adique de C . On a la suite exacte

$$I/I^2 \xrightarrow{\delta} \Omega_{C/A} \otimes_C B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

D'après le critère jacobien de lissité formelle, il suffit de prouver que δ est formellement inversible à gauche. Soient $C' = C/MC$, $B' = B$ et $I' = \ker v \otimes_A (A/M)$. On a alors le diagramme commutatif suivant :

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{\delta_1} & \Omega_{C/A} \otimes_C B' & \longrightarrow & \Omega_{B/A} \otimes_B B' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ I'/I & \xrightarrow{\delta'} & \Omega_{C'/k} \otimes_{C'} B' & \longrightarrow & \Omega_{B'/k} & \longrightarrow & 0 \end{array}$$

où $k = A/M$. On sait que g et h sont des isomorphismes et du fait que $\text{Tor}_1^A(B, A/M) = 0$, on déduit que f est aussi un isomorphisme. Puisque B' est une k -algèbre formellement lisse, le morphisme δ' est formellement inversible à gauche. Du lemme on déduit que les topologies sur I/I^2 et I'/I'^2 sont les topologies N -adiques. On peut donc appliquer la proposition 19.1.9 de EGA. O_{IV} et on obtient ainsi que δ est formellement inversible à gauche.

Pour démontrer l'implication $a) \Rightarrow b)$ du théorème, nous appliquerons le théorème de structure des anneaux locaux noethériens complets ce qui se démontre en utilisant l'implication $b) \rightarrow a)$ pour montrer qu'un p -anneau de Cohen est une $Z_{(p)}$ -algèbre formellement lisse.

Démonstration de l'implication $a) \Rightarrow b)$ du théorème. Puisque $u : A \rightarrow B$ est formellement lisse, on déduit que $u \otimes k : k \rightarrow B \otimes_A k$ est un morphisme formellement lisse. Pour démontrer que B est un A -modul plat, il suffit de prouver que le complété $\hat{B}$ de B est un $\hat{A}$ -module plat, où $\hat{A}$ est le complété de A dans la topologie M -adique. Donc on peut supposer que A et B sont complets (pour les topologies radicielles). Soit p la caractéristique de k . Soit W un p -anneau de Cohen dont le corps résiduel soit isomorphe à k si A ne contient pas un corps et $W = k$ au cas contraire. De même soit W' un p -anneau de Cohen dont le corps résiduel soit isomorphe au corps résiduel K de B si B ne contient pas un corps et $W = K$ au cas contraire. Soit W_0 l'anneau local premier $Z_{(p)}$ si W est un p -anneau de Cohen et le corps premier contenu dans W si W est un corps. D'après le théorème de structure des anneaux locaux noethériens complets on déduit qu'il existe un morphisme surjectif d'anneaux $q' : W'[[X_1, \dots, X_n]] \rightarrow B$ et puisque W est une W_0 -algèbre formellement lisse, il existe un morphisme d'anneaux $u_1 : W \rightarrow W'[[X_1, \dots, X_n]]$, tel que le diagramme suivant soit commutatif

$$\begin{array}{ccc} W & \xrightarrow{u_1} & W'[[X_1, \dots, X_n]] \\ r' \downarrow & & \downarrow q' \\ A & \xrightarrow{u} & B \end{array}$$

où r' est un morphisme construit du fait que W est une W_0 -algèbre formellement lisse. Soient $x_1, \dots, x_m$ un système de générateurs de l'idéal maximal de A et $r : W[[Y_1, \dots, Y_m]] \rightarrow A$ le morphisme d'anneaux prolongeant r' tel que $r(Y_i) = x_i$, $i = 1, \dots, m$. Soit $q : W[[X_1, \dots, X_n, Y_1, \dots, Y_m]] \rightarrow B$ le morphisme d'anneaux prolongeant q' tel que $q(Y_i) = u(x_i)$, $i = 1, 2, \dots, m$. On peut prolonger le morphisme u_1 au morphisme $u' : W[[Y_1, \dots, Y_m]] \rightarrow W'[[X_1, \dots, X_n, Y_1, \dots, Y_m]]$ tel que $u'(Y_i) = Y_i$, $i = 1, \dots, m$. Alors on obtient le diagramme commutatif

$$\begin{array}{ccc} A' = W[[Y_1, \dots, Y_m]] & \xrightarrow{u'} & W'[[X_1, \dots, X_n, Y_1, \dots, Y_m]] = B' \\ r \downarrow & & \downarrow q \\ A & \xrightarrow{u} & B \end{array}$$

Le morphisme u' est plat, puisque les éléments $u'(Y_1), \dots, u'(Y_m)$ de B' est une partie d'un système minimal de générateurs de l'idéal maximal de B' . Alors l'anneau $B' \otimes_A A$ est un anneau complet (il est un anneau quotient de B') et B est une A -algèbre formellement lisse. Il en résulte qu'il existe un morphisme de A -algèbres $v: B \rightarrow C$ tel que $q''v = 1_B$. Puisque v et q'' sont des morphismes de A -algèbres, ils sont aussi des morphismes de A -modules, donc B est un facteur direct de C qui est un A -module plat.

On peut voir que l'implication $a) \Rightarrow b)$ du théorème n'est pas vraie si l'anneau A n'est pas noethérien. En effet, soit k un corps et A un anneau de valuation de dimension 1, de corps résiduel isomorphe à k et tel que la valuation de A n'est pas discrète. Soit M l'idéal maximal de A . Alors $M = M^2$, d'où on déduit que k est une A -algèbre formellement lisse.

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Faculté de Mathématique
Université de Bucarest

1. Introduction

In a previous paper [2] the author defined the generalised function of two variables as follow

$$S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; \quad b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; \quad d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; \quad f_1, \dots, f_{q_3} \end{array} \middle| x, y, \right] \\
 = \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j + s) \prod_{j=1}^{n_2} \Gamma(d_j - s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + s + t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j - s)} \times \\
 \times \frac{\prod_{j=1}^{m_3} \Gamma(1 - e_j - t) \prod_{j=1}^{n_3} \Gamma(f_j - t)}{\prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j + s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j - t) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + t)} x^s y^t ds dt, \quad (1)$$

where L_1 and L_2 are two suitable contours and the positive integers $p_1, p_2, p_3, q_1, q_2, q_3, m_1, m_2, m_3, n_2$ and n_3 satisfy the following inequalities

$$\begin{aligned}
 & q_2 > 1, \quad q_3 > 1, \quad p_1 > 0, \quad q_1 > 0, \quad 0 \leq m_1 \leq p_1, \quad 0 \leq m_2 \leq p_2 \\
 & 0 \leq n_2 \leq q_2, \quad 0 \leq m_3 \leq p_3, \quad 0 \leq n_3 \leq q_3, \quad p_1 + p_2 \leq q_1 + q_2 \quad \text{and} \\
 & p_1 + p_3 \leq q_1 + q_3.
 \end{aligned}$$

The values $x = 0$ and $y = 0$ are excluded.

Its convergence, evaluation and analytic continuation have been discussed in my previous paper [2]. The object of this paper is to give the elementary properties and the recurrence relations for the generalised function of two variables.

1.1. *First property* — If one of the $a_j (j = 1, 2, \dots, m_1)$ is equal to one of $b_j (j = 1, 2, \dots, q_1)$ or one of $c_j (j = 1, 2, \dots, m_2)$, $c_j (j = m_2 + 1, \dots, p_2)$, $e_j (j = 1, 2, \dots, m_3)$ and $e_j (j = m_3 + 1, \dots, p_3)$ is equal to one of $d_j (j = n_2 + 1, \dots, q_2)$, $d_j (j = 1, 2, \dots, n_2)$, $f_j (j = n_3 + 1, \dots, q_3)$ and $f_j (j = 1, 2, \dots, n_3)$ respectively, then the generalised function of two variables reduces to one of the lower order i.e. m_1, p_1 and q_1 (or, m_2, p_2 and q_2 ; n_2, p_2 and q_2 ; m_3, p_3 and q_3 ; n_3, p_3 and q_3 respectively) decrease by unity. For example

$$S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; a_1, b_2, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, d_{q_2-1} \quad c_1 \\ e_1, \dots, e_{p_3-2}, f_1; f_1, \dots, f_{q_3} \end{array} \middle| x, y \right] =$$

$$= S \left[\begin{array}{c} \left[\begin{array}{cc} m_1 - 1, & 0 \\ p_1 - m_1, & q_1 - 1 \end{array} \right] \\ \left(\begin{array}{cc} m_2 - 1, & n_2 \\ p_2 - m_2, & q_2 - n_2 - 1 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 - 1 \\ p_3 - m_3 - 1, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_2, \dots, a_{p_1}; b_2, \dots, b_{q_1} \\ c_2, \dots, c_{p_2}; d_1, \dots, d_{q_2-1} \\ e_1, \dots, e_{p_3-1}; f_2, \dots, f_{q_3} \end{array} \middle| x, y \right]$$

1.2. *Second property* : — If we multiply the generalised function of two by variables $x^\lambda y^\mu$ then each parameters associated with $s + t (a_1 \dots a_{p_1}; b_1 \dots b_{q_1})$ decrease by the amount $\lambda + \mu$ and the parameters associated with $s(c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2})$ and $t(e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3})$ increase by the amount λ and μ respectively. Thus

$$x^\lambda y^\mu S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \middle| x, y \right] =$$

$$= S \left[\begin{array}{cc|c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] & a_1-\lambda-\mu, \dots, a_{p_1}-\lambda-\mu; b_1-\lambda-\mu, \dots, & \\ & \dots, b_{q_1}-\lambda-\mu & \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{array} \right) & c_1+\lambda, \dots, c_{p_2}+\lambda; d_1+\lambda, \dots, d_{q_2}+\lambda & x, y \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{array} \right) & e_1+\mu, \dots, e_{p_3}+\mu; f_1+\mu, \dots, f_{q_3}+\mu & \end{array} \right] \quad (3)$$

To prove (3), we write the contour integral (1) for the generalised function of two variables on the right side of (3). We get

$$\begin{aligned} & \frac{i}{(2\pi i)} \int_{L_2} \int_{L_1} \frac{\prod_{j=1}^{m_1} \Gamma(a_j - \lambda - \mu + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j - \lambda + s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j + \lambda + \mu - s - t) \prod_{j=1}^{q_1} \Gamma(b_j - \lambda - \mu + s + t)} \times \\ & \times \frac{\prod_{j=1}^{p_2} \Gamma(d_j + \lambda - s) \prod_{j=1}^{m_3} \Gamma(1 - e_j - \mu + t) \prod_{j=1}^{n_3} \Gamma(f_j + \mu + t)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j + \lambda - s) \prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j - \lambda + s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j + \mu - t)} \times \\ & \times \frac{1}{\prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j - \mu - t)} x^s y^t ds dt. \end{aligned}$$

On replacing s and t by $s + \lambda$ and $t + \mu$ respectively, the right side becomes

$$\begin{aligned} & x^\lambda y^\mu \frac{1}{(2\pi i)} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j + s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + s + t)} \times \\ & \times \frac{\prod_{j=1}^{n_2} \Gamma(d_j - s) \prod_{j=1}^{m_3} \Gamma(1 - e_j + t) \prod_{j=1}^{n_3} \Gamma(f_j - t)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j - s) \prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j + s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j - t) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + t)} x^s y^t ds dt \end{aligned}$$

On interpreting the above result with the help of (1), we get the left side of the second property.

1.3. *Third property* : — If m and n are positive integers, then

$$S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2+1 \\ p_2-m_2+1, & q_2-n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3-1 \\ p_3-m_3+1, & q_3-n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_1}, 1-m; 1, d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}, 1-n, 1, f_1, \dots, f_{q_3} \end{array} \middle| x, y \right] =$$

$$(-1)^{m+n} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2+1, & n_2 \\ p_2-m_2, & q_2-n_2+1 \end{array} \right) \\ \left(\begin{array}{cc} m_3+1, & n_3 \\ p_3-m_3, & q_3-n_3+1 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ 1-m, c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2}, 1 \\ 1-n, e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3}, 1 \end{array} \middle| x, y \right] \quad (4)$$

To prove (4) we write the contour integral (1) for the right side, it then becomes.

$$(-1)^{m+n} \frac{1}{(2\pi i)} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j+s+t) \prod_{j=1}^{m_2} \Gamma(1-c_j+s) \prod_{j=1}^{n_2} \Gamma(d_j-s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1-a_j-s-t) \prod_{j=1}^{q_1} \Gamma(b_j+s+t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j-s)} \times$$

$$\times \frac{\prod_{j=1}^{m_3} \Gamma(1-e_j+t) \prod_{j=1}^{n_3} \Gamma(f_j-t) \Gamma(m+s) x^s y^t}{\prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-t) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+t) \Gamma(s) \Gamma(t)} ds dt.$$

Now using the relation [1, p. 3]

$$\frac{\Gamma(1-s)}{\Gamma(1-r-s)} = (-1)^r \frac{\Gamma(r+s)}{\Gamma(s)} \quad (5)$$

where r is a positive integer, it takes the form

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j + s) \prod_{j=1}^{n_2} \Gamma(d_j - s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + s + t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j - s)} \times$$

$$\times \frac{\Gamma(1-s) \prod_{j=1}^{m_3} \Gamma(1 - e_j + t) \prod_{j=1}^{n_3} \Gamma(f_j - t) \Gamma(1-t) x^s y^t ds dt}{\Gamma(1-m-s) \prod_{j=m_3+1}^{q_3} \Gamma(e_j - t) \Gamma(1-n-t) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j + s) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j + t)}$$

on interpreting the above result with the help of (1), we get the left side of the third property.

1.4. *Fourth property*: -- If m is a positive integer, then

$$S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; \quad b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; \quad d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; \quad f_1, \dots, f_{q_3} \end{array} \right] x, y$$

$$= (2\pi)^{(1-m)} \left(m_1 + m_2 + n_2 - m_3 - n_2 - \frac{1}{2} p_1 - \frac{1}{2} q_1 - \frac{1}{2} p_2 - \frac{1}{2} q_2 - \frac{1}{2} p_3 - \frac{1}{2} q_3 \right)$$

$$(m) \sum_{j=1}^{p_1} a_j + \sum_{j=1}^{q_2} d_j + \sum_{j=1}^{q_3} f_j - \sum_{j=1}^{p_2} c_j - \sum_{j=1}^{p_3} e_j + \frac{1}{2} (p_2 + p_3 + q_1 - p_1 - q_2 - q_3 - 4)$$

$$S \left[\begin{array}{c} \left[\begin{array}{cc} mm_1, & 0 \\ m(p_1 - m_1), & mq_1 \end{array} \right] \\ \left(\begin{array}{cc} m m_2, & m n_2 \\ m(p_2 - m_2), & m(q_2 - n_2) \end{array} \right) \\ \left(\begin{array}{cc} m m_3, & m n_3 \\ m(p_3 - m_3), & m(q_3 - n_3) \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; \quad \beta_1, \dots, \beta_{q_1} \\ \gamma_1, \dots, \gamma_{p_2} \quad \delta_1, \dots, \delta_{q_2} \\ \varphi_1, \dots, \varphi_{p_3} \quad \psi_1, \dots, \psi_{q_3} \end{array} \right] x^m(m)^{m(p_1 + p_2 - q_1 - q_2)} y^m(m)^{m(p_1 + p_3 - q_1 - q_3)}$$

(6)

$$\text{where } \alpha_j = \frac{a_j}{m}, \dots, \frac{a_j+m-1}{m}, j = 1, 2, \dots, p_1,$$

$$\beta_j = \frac{b_j}{m}, \dots, \frac{b_j+m-1}{m}, j = 1, 2, \dots, q_1$$

$$\gamma_j = \frac{c_j}{m}, \dots, \frac{c_j+m-1}{m}, j = 1, 2, \dots, p_2,$$

$$\delta_j = \frac{d_j}{m}, \dots, \frac{d_j+m-1}{m}, j = 1, 2, \dots, q_2,$$

$$\varphi_j = e_j, \dots, \frac{e_j+m-1}{m}, j = 1, 2, \dots, p_3,$$

$$\psi_j = f_j, \dots, \frac{f_j+m-1}{m}, j = 1, 2, \dots, q_3.$$

To prove (6), we write the contour integral (1) for the generalised function of two variables on the left side of (6) and replacing s and t by ms and mt respectively, it then becomes

$$\frac{m^2}{(2\pi i)^2} \iint_{L_1 L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j+ms+mt) \prod_{j=1}^{m_2} \Gamma(1-c_j+ms) \prod_{j=1}^{n_2} \Gamma(d_j-ms)}{\prod_{j=m_1+1}^{p_1} \Gamma(1-a_j-ms-mt) \prod_{j=1}^{q_1} \Gamma(b_j+ms+mt) \prod_{j=m_2+1}^{p_2} \Gamma(c_j-ms)} \\ \frac{\prod_{j=1}^{m_3} \Gamma(1-e_j+mt) \prod_{j=1}^{n_3} \Gamma(f_j-tm) x^{ms} y^{mt}}{\prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+ms) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-mt) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+mt)} ds dt$$

Using the multiplication formula for the Gamma function [1, p.4]

$$\Gamma(mz) = (2\pi)^{\frac{1}{2} - \frac{1}{2}m} (m)^{mz - \frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right). \quad (7)$$

where m is a positive integer, it takes the form

$$\begin{aligned}
 & (2\pi)^{(1-m)} \left(m_1 + m_2 + m_3 + n_2 + n_3 - \frac{1}{2} p_1 - \frac{1}{2} p_2 - \frac{1}{2} p_3 - \frac{1}{2} q_1 - \frac{1}{2} q_2 - \frac{1}{2} q_3 \right) \\
 & (m)^{j=1} \sum_{j=1}^{p_1} a_j + \sum_{j=1}^{q_2} d_j + \sum_{j=1}^{q_3} f_j - \sum_{j=1}^{q_1} b_j - \sum_{j=1}^{p_2} c_j - \sum_{j=1}^{p_3} e_j + \frac{1}{2} (p_2 + p_3 + q_1 - p_1 - q_2 - q_3 + 4) \\
 & \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \prod_{k=0}^{m-1} \Gamma\left(\frac{a_j + k}{m} + s + t\right) \prod_{j=1}^{m_2} \prod_{k=0}^{m-1} \Gamma\left(\frac{1+k-c_j}{m} + s\right)}{\prod_{j=m_1+1}^{p_1} \prod_{k=0}^{m-1} \Gamma\left(\frac{k+1-a_j}{m} - s - t\right) \prod_{j=1}^{q_1} \prod_{k=0}^{m-1} \Gamma\left(\frac{k+b_j}{m} + s + t\right)} \times \\
 & \times \frac{\prod_{j=1}^{n_2} \prod_{k=0}^{m-1} \Gamma\left(\frac{k+d_j}{m} - s\right) \prod_{j=1}^{m_3} \prod_{k=0}^{m-1} \Gamma\left(\frac{1-e_j+k}{m} + t\right) \prod_{j=1}^{n_3} \prod_{k=0}^{m-1} \Gamma\left(\frac{f_j+k}{m} - t\right)}{\prod_{j=m_2+1}^{p_2} \prod_{k=0}^{m-1} \Gamma\left(\frac{c_j+k}{m} - s\right) \prod_{j=n_2+1}^{q_2} \prod_{k=0}^{m-1} \Gamma\left(\frac{1-d_j+k}{m} + s\right) \prod_{j=m_3+1}^{p_3} \prod_{k=0}^{m-1} \Gamma\left(\frac{e_j+k}{m} - t\right)} \times \\
 & \frac{(m)^{m(p_1+p_2-q_1-q_2)s+m(p_1+p_3-q_1-q_3)t} x^{ms} y^{mt}}{\prod_{j=n_3+1}^{q_3} \prod_{k=0}^{m-1} \Gamma\left(\frac{1-f_j+k}{m} + t\right)} ds dt.
 \end{aligned}$$

On interpreting the above result with the help of (1), we get the right side of the fourth property.

1.5. *Fifth property* : — If p and q are positive integers, and $x \neq 0$ and $y \neq 0$ then

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \right] \left[\begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \right] x, y \Bigg] =$$

$$x^{-p}y^{-q} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2+1, & n_2 \\ p_2 - m_2, & q_2 - n_2 + 1 \end{array} \right) \\ \left(\begin{array}{cc} m_3+1, & n_3 \\ p_3 - m_3, & q_3 - n_3 + 1 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ 0, c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2}, p \\ 0, e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3}, q \end{array} \right] x, y,$$

If $x \neq 0$ and $y \neq 0$, then we have

$$\begin{aligned} \frac{\partial^{p+q}}{\partial x^p \partial y^q} (x^s y^t) &= s(s-1) \dots (s-p+1) t(t-1) \dots (t-q+1) x^{s-p} y^{t-q} = \\ &= \frac{\Gamma(1+s) \Gamma(1+t)}{\Gamma(1-p+s) \Gamma(1-q+t)} x^{s-p} y^{t-q} \end{aligned} \quad (9)$$

It follows from the definition (1) of our generalised function and making use of (9) the left side of (8) is equal to

$$\begin{aligned} \frac{1}{(2\pi i)^2} &= \iint_{L_1 L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j + s) \prod_{j=1}^{n_2} \Gamma(d_j - s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + s + t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j - s)} \\ &\quad \frac{\prod_{j=1}^{m_3} \Gamma(1 - e_j + t) \prod_{j=1}^{n_3} \Gamma(f_j - t) \Gamma(1+s) \Gamma(1+t) x^{s-p} y^{t-q}}{\prod_{j=s+1}^{q_2} \Gamma(1 - d_j + s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j - t) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + t) \Gamma(1-p+s) \Gamma(1-q+t)} \times \\ &\quad \times ds dt. \end{aligned}$$

Now replacing s and t by $s + p$ and $t + q$ respectively, it becomes

$$\frac{1}{(2\pi i)^2} \iint_{L_1 L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + p + q + s + t) \prod_{j=1}^{m_2} \Gamma(1 - c_j + p + s) \prod_{j=1}^{n_2} \Gamma(d_j - s - p)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - p - q - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + p + q + s + t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j - s - p)} \times$$

$$\times \frac{\prod_{j=1}^{m_3} \Gamma(1-e_j+t+q) \prod_{j=1}^{n_3} \Gamma(f_j-t-q) \Gamma(1+s+p) \Gamma(1+t+q)}{\prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+s+p) \prod_{j=m_3+j}^{p_2} \Gamma(e_j-t-q) \prod_{1=n_3+1}^{q_3} \Gamma(1-f_j+t+q)} \times$$

$$\frac{x^s y^t}{\Gamma(1+s) \Gamma(1+t)} ds dt.$$

(8) follows immediately on interpreting the above result with the help (1) and using (3).

1.6. *Sixth property*: — If p and q are positive integers, $x \neq 0$ and $y \neq 0$, then

$$x^p y^q \frac{\partial^{p+q}}{\partial x^p \partial y^q} S \left[\begin{matrix} \left[\begin{matrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{matrix} \right] \\ \left(\begin{matrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{matrix} \right) \\ \left(\begin{matrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{matrix} \right] \frac{1}{x}, \frac{1}{y} \right] =$$

$$= (-1)^{p+q} S \left[\begin{matrix} \left[\begin{matrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{matrix} \right] \\ \left(\begin{matrix} m_2+1, & n_2 \\ p_2-m_2, & q_2-n_2+1 \end{matrix} \right) \\ \left(\begin{matrix} m_3+1, & n_3 \\ p_3-m_3, & q_3-n_3+1 \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ 1-p, c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2}, 1 \\ 1-q, e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3}, 1 \end{matrix} \right] \frac{1}{x}, \frac{1}{y} \right] \quad (10)$$

(10) can be proved on the similar lines as (8) on using the result

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} (x^{-s} y^{-t}) = \frac{(-1)^{p+q} \Gamma(p+s) \Gamma(q+t)}{\Gamma(s) \Gamma(t)} x^{-p-s} y^{-q-t}, \quad (11)$$

instead of (9).

1.7. *Seventh property*: If $x \neq 0$ and $y \neq 0$ then

$$\begin{aligned}
 & \text{(a)} \quad \frac{\partial}{\partial x \partial y} x^{-d_1} y^{-f_1} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1}, \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \right] x, y = \\
 & = x^{-1-d_1} y^{-1-f_1} S \left[\begin{array}{c} \left(\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right) \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; 1 + d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; 1 + f_1, \dots, f_{q_3} \end{array} \right] x, y
 \end{aligned} \tag{12}$$

where $q_2, q_3, n_2, n_3 \geq 1$.

$$\begin{aligned}
 & \text{(b)} \quad \frac{\partial}{\partial x \partial y} x^{-d_{q_3}} y^{-f_{q_3}} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_3 - m_3, & q_2 - n_2 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_1}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_2}; f_1, \dots, f_{q_3} \end{array} \right] x, y = \\
 & = x^{-1-d_{q_3}} y^{-1-f_{q_3}} S \left[\begin{array}{c} \left(\begin{array}{cc} m_1, & 0 \\ p_1 - m_1, & q_1 \end{array} \right) \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{array} \right) \\ \left[\begin{array}{cc} m_3, & n_3 \\ p^3 - m^3, & q_3 - n_3 \end{array} \right] \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} + 1 \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} + 1 \end{array} \right] x, y
 \end{aligned} \tag{3}$$

where $n_2 < q_2$, $n_3 < q_3$; $q_2, q_3 \geq 1$.

$$\begin{aligned}
 & \text{(c)} \quad \frac{\partial}{\partial x \partial y} x^{e_1-1} y^{e_1-1} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \right] \frac{1}{x}, \frac{1}{y} = \\
 & = x^{e_1-2} y^{e_1-2} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1-1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1-1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \right] \frac{1}{x}, \frac{1}{y} \\
 & \hspace{15cm} (14)
 \end{aligned}$$

where $m_2 \geq 1$, $m_3 \geq 1$.

$$\begin{aligned}
 & \text{(d)} \quad \frac{\partial}{\partial x \partial y} x^{e_{p_2}-1} y^{e_{p_3}-1} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{array} \right] \frac{1}{x}, \frac{1}{y} \\
 & = x^{e_{p_2}-2} y^{e_{p_3}-2} S \left[\begin{array}{c} \left[\begin{array}{cc} m_1, & 0 \\ p_1-m_1, & q_1 \end{array} \right] \\ \left(\begin{array}{cc} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{array} \right) \\ \left(\begin{array}{cc} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{array} \right) \end{array} \middle| \begin{array}{c} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}-1; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}-1; f_1, \dots, f_{q_3} \end{array} \right] \frac{1}{x}, \frac{1}{y} \\
 & \hspace{15cm} (15)
 \end{aligned}$$

where $m_2 < p_2$, $m_3 < p_3$; $p_2, p_3 \geq 1$.

Similar formulæ can be written down for the parameters $a_1, \dots, a_{p_1}$ and $b_1, \dots, b_{q_1}$. We give the proof of (12) only. Other results can be proved in the similar way.

Now

$$\begin{aligned} & \frac{\partial}{\partial x \partial y} x^{-d_1} y^{-f_1} S \left[\begin{matrix} \left[\begin{matrix} m_1, & 0 \\ p_1 - m_1, & q_1 \end{matrix} \right] \\ \left(\begin{matrix} m_2, & n_2 \\ p_2 - m_2, & q_2 - n_2 \end{matrix} \right) \\ \left(\begin{matrix} m_3, & n_3 \\ p_3 - m_3, & q_3 - n_3 \end{matrix} \right) \end{matrix} \middle| \begin{matrix} a_1, \dots, a_{p_1}; b_1, \dots, b_{q_1} \\ c_1, \dots, c_{p_2}; d_1, \dots, d_{q_2} \\ e_1, \dots, e_{p_3}; f_1, \dots, f_{q_3} \end{matrix} \right] y, y \\ &= \frac{1}{(2\pi i)^2} x^{-1-d_1} y^{-1-f_1} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j + s + t) \prod_{j=2}^{n_2} \Gamma(d_j - s)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - s - t) \prod_{j=1}^{q_1} \Gamma(b_j + s + t)} \times \\ & \times \frac{\prod_{j=1}^{m_2} \Gamma(1 - c_j + s) \prod_{j=1}^{m_3} \Gamma(1 - e_j + t) \prod_{j=2}^{n_3} \Gamma(f_j - t)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j - s) \prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j + s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j - t) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + t)} \\ & \times \Gamma(1 + d_1 - s) \Gamma(1 + f_1 - t) x^s y^t ds dt. \quad \dots(16) \end{aligned}$$

On interpreting the result (16) with the help of (1), we get (12). The other aspects of the generalised function of two variables will be discussed in the next paper.

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Department of Mathematics
University of Ife,
Ile-Ife, Nigeria

PARTIAL POINT DERIVATIONS ON LIPSCHITZ ALGEBRAS

LILIANA STOICA

In this note we show that the study of the partial point derivations is not a trivial consequence of the general theory of point derivations (in the sense of [1], [2], [3]). We prove the existence of the nontrivial partial point derivations at each nonisolated point for the Lipschitz functions of several variables and we give some properties of these derivations.

1. Preliminaries Let (X, d) be a metric space and let $\text{Lip}(X, d)$ be the Banach algebra of all bounded Lipschitz functions on (X, d) . According to [1] one characterizes the point derivations on $\text{Lip}(X, d)$ at each nonisolated point x_0 of (X, d) in terms of weak*, cluster point of certain sequences $\{\Phi_n\}$ of bounded linear functionals on $\text{Lip}(X, d)$ where Φ_n is defined by

$$1.1. \quad \Phi_n(f) = \frac{f(x_n) - f(x'_n)}{d(x_n, x'_n)}$$

associated to the sequences (x_n, x'_n) of pairs of points from X such that $x_n \neq x'_n$ and $x_n \rightarrow x_0, x'_n \rightarrow x_0$.

Simple examples show, that it is necessary to consider two sequences $\{x_n\}, \{x'_n\}$ such that $x_n \rightarrow x_0, x'_n \rightarrow x_0$ since the sequences of functionals defined by only a single sequence of points converging to x_0 are inadequate to obtain all of point derivations at the point $x_0 \in X$.

Knowing the properties of partial derivations in the classic sense and the axiomatization of the derivation in the sense of [1], [2], [3] the partial derivations on the algebra A of functions in the point (x_0, y_0) , must be linear functionals $\partial_1^1, \partial_2^1$, on A , which satisfy the product rule

$$(1.2) \quad \partial_i^1(f \cdot g) = f(x_0, y_0) \partial_i^1 g + g(x_0, y_0) \partial_i^1 f, \quad i = 1, 2$$

$$f, g \in A$$

2. Now we give the construction of the partial point derivations on the algebra A , of Lipschitz functions of two variables. Let $(X, \rho), (Y, \sigma)$, be two metric spaces. As usually, we say $f : X \times Y \rightarrow C$ is a Lipschitz function if there exist $M_1 > 0, M_2 > 0$ such that.

$$2.1) \quad |f(x, y) - f(x', y')| \leq M_1 \rho(x, x_0) + M_2 \sigma(y, y_0) \text{ for all } x, x' \in X$$

and $y, y' \in Y$

Proposition 2.1: The condition (2.1) is equivalent to the following two conditions :

1) f is a Lipschitz function relative to x , uniformly relative to y , i.e., there exists a constant $L_1 > 0$ such that

$$|f(x, y) - f(x', y)| < L_1 \rho(x, x') \text{ for all } x, x' \in X \text{ and } y \in Y$$

2) f is a Lipschitz function relative to y , uniformly relative to x , i.e., there exists $L_2 > 0$ such that

$$|f(x, y) - f(x, y')| < L_2 \sigma(y, y') \text{ for all } x \in X \text{ and } y, y' \in Y$$

Corollary 2.1: The functionals, $\|\cdot\|_x$ $\|\cdot\|_y$ on A

$$\|f\|_x = \sup \left\{ \frac{|f(x, y) - f(x', y)|}{\rho(x, x')} ; \begin{array}{l} (x, y) \in X \times Y \\ (x', y) \in X \times Y \end{array} \begin{array}{l} x \neq x' \\ y \neq y' \end{array} \right\}$$

and

$$\|f\|_y = \sup \left\{ \frac{|f(x, y) - f(x, y')|}{\sigma(y, y')} ; \begin{array}{l} (x, y) \in X \times Y \\ (x, y') \in X \times Y \end{array} \begin{array}{l} x \neq x' \\ y \neq y' \end{array} \right\}$$

are semi-norms on the algebra A .

For a complex-valued function f defined on $X \times Y$ which is bounded on $X \times Y$, the sup. norm. $\|f\|_\infty$ is defined by

$$\|f\|_\infty = \sup \{ |f(x, y)| ; (x, y) \in X \times Y \}$$

Proposition 2.2: The functionals $\|\cdot\|^1$ and $\|\cdot\|^2$ on A given by

$$(2.2) \quad \|f\|^1 = \|f\|_x + \|f\|_\infty \quad \|f\|^2 = \|f\|_y + \|f\|_\infty$$

are Banach algebras norms on the algebra A .

Let $(x_0, y_0) \in X \times Y$. We assume (x_0, y_0) is not isolated in the topology product.

$$\text{Let } W^1 = \left\{ (z, z') ; \begin{array}{l} z = (x, y) \in X \times Y \\ z' = (x', y) \in X \times Y \end{array} \begin{array}{l} x \neq x' \\ y \neq y' \end{array} \right\} \text{ and}$$

$$W^2 = \left\{ (z, z') ; \begin{array}{l} z = (x, y) \in X \times Y \\ z' = (x, y') \in X \times Y \end{array} \begin{array}{l} x \neq x' \\ y \neq y' \end{array} \right\}.$$

Let $W^1_{(x_0, y_0)}$ denote the set of all sequences $\{(x_n, y_n), (x'_n, y_n)\}$ from W^1 , which converge to (x_0, y_0) and $W^2_{(x_0, y_0)}$ denotes set of all sequences $\{(x_n, y_n), (x_n, y'_n)\}$ from W^2 which converge to (x_0, y_0) . To each element of

$W_{(x_0, y_0)}^1$ we associate the sequence ψ_n of bounded linear functionals on A , where f_n is defined by

$$(2.3) \quad \psi_n(f) = \frac{f'(x_n, y_n) - f(x'_n, y'_n)}{\rho(x_n, x'_n)}, \quad f \in A$$

and similarly, to each element of $W_{(x_0, y_0)}^2$ we associate the sequence χ_n defined by

$$\chi_n(f) = \frac{f(x_n, y_n) - f(x_n, y'_n)}{\sigma(y_n, y'_n)}, \quad f \in A$$

Theorem 2.1. The collection of all weak * cluster points of the set of all sequences $\{\chi_n\}$, defined by (2.3) such as $\{(x_n, y_n), (x'_n, y'_n)\}$ varies over $W_{(x_0, y_0)}^1$, is nonempty and any weak * cluster point is a partial point derivation ∂_1^1 at (x_0, y_0)

Proof Let A^* be the dual space and $\|\cdot\|$ denotes the norm in the dual.

Clearly $|\psi_n(f)| < \|f\|_x$

We have $\|\psi_n\|^* = \sup \{|\psi_n(f)| : \|f\|_x \leq 1\}$. That is, the sequence $\{\psi_n\}$ lies in the unit sphere S^* of the dual space A^* . Let $\psi_{(x_0, y_0)}$ denote the collection of all weak * cluster point of the set of all sequence $\{\psi_n\}$. We have $\psi_{(x_0, y_0)} \subset S^*$ and $\psi_{(x_0, y_0)} \neq \emptyset$. Suppose $\partial_1^1 \in \psi_{(x_0, y_0)}$. Let $\{\psi_{n_j}\}$ be a subset of $\{\psi_n\}$ which converges to ∂_1^1 in the weak * topology. Let $\{(x_{n_j}, y_{n_j}), (x'_{n_j}, y'_{n_j})\}$ be the corresponding subset of $\{(x_n, y_n), (x'_n, y'_n)\}$. Then for any f, g in A .

$$\partial_1^1(f, g) = \lim \psi_{n_j}(f, g) = \lim \frac{(fg)(x_{n_j}, y_{n_j}) - (fg)(x'_{n_j}, y'_{n_j})}{\rho(x_{n_j}, x'_{n_j})} =$$

$$= g(x_0, y_0) \lim \psi_{n_j}(f) + f(x_0, y_0) \lim \psi_{n_j}(g) =$$

$$= g(x_0, y_0) \partial_1^1 f + f(x_0, y_0) \partial_1^1 g.$$

Hence ∂_1^1 is a partial point derivation on at (x_0, y_0) . Similarly, any weak * cluster point of the set of all sequence $\{\psi_n\}$ is a partial point ∂_2^1 derivation at (x_0, y_0) .

Theorem 2.5: There exist nontrivial partial point derivations on A relative to each variable at each nonisolated point of $X \times Y$

Proof: If (x_0, y_0) is a nonisolated point of $X \times Y$, there exists a sequence $\{Z_n\}$, $Z_n = (x_n, y_n)$ and $Z_n \rightarrow (x_0, y_0)$. Then the sequence

$\{(x_n, y_n), (x_0, y_0)\} \in W_{(x_0, y_0)}^1$ and the sequence $\{\psi_n\}$ is given by (2.3) with $x'_n = x_0$, for all n . Let us define g on $X \times Y$ by

$$g(t, \tau) = \rho(x_0, t) + \sigma(y_0, \tau) \text{ where } (t, \tau) \in X \times Y$$

Clearly $g \in A$ if the diameters of (X, d) and (Y, σ) are finite. If one of the diameters is infinite, then by truncating the function g at t the function $g_1(t, \tau) = \min [g(t, \tau), 1]$ belongs to A , as [4].

$$\text{Thus } \psi_n(g) = \frac{g(x_n, y_n) - g(x_0, x_n)}{\rho(x_n, x_0)} = 1.$$

If ∂_1^1 is any weak * cluster point of $\{\psi_n\}$, then $\partial_1^1 g = 1$. Hence ∂_1^1 is a nontrivial partial point derivation at (x_0, y_0) . Similarly, the sequence $\{(x_n, y_n), (x_n, y_0)\} \in W_{(x_0, y_0)}^2$ and $\psi_n(g) = 1$. If ∂_2^1 is any weak * cluster point of $\{\chi_n\}$, then $\partial_2^1 g = 1$ and ∂_2^1 is a nontrivial partial point derivation at (x_0, y_0) .

Proposition 2.3: For each $(x_0, y_0) \in X \times Y$, the spaces D_1^1 and D_2^1 of partial point derivations ∂_1^1 and ∂_2^1 are the linear subspaces of A^* closed in the weak * topology.

Remain valide all properties of point derivation from [1] for partial point derivations.

In the futher papers we will use the partial point derivations in order to study the higher partial point derivation on A .

Remark. Apparently, we may obtain the notion of partial point derivation for the fnctions of several variables (we considered the case of two variables (by applying the method of [1] to the functions of one variable obtained by fixing the others, to certain values. This strict analogy to the notion of partial derivation of the classic analysis does not permit to describe all the partial point derivations.

This can be seen by considering the function $f: [0,1]^2 \rightarrow R$

$$f(x,y) = \begin{cases} \frac{xy}{x^3 + y^3} & x \neq 0 \text{ or } y \neq 0 \\ 0 & x = 0 \text{ and } y = 0. \end{cases}$$

We observe that $f \in \text{Lip } (X \times Y)$. The partial derivations at the point $(0,0)$ in the classic sense are null. Similarly the partial point derivations in the sense [1] are null. If we consider the sequences of the form

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}, \quad x'_n = \frac{2}{n} \quad \text{then} \quad \partial_1^1(0,0) \neq 0$$

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*Mathematical Department,
University of Craiova.*

ON A CONDITION WHAT IMPLIES THE HIPOELLIPTICITY OF PSEUDODIFFERENTIAL OPERATORS

VIOREL ZLATE

I. Introductory part

In this text, we use the notations usual from the pseudodifferential operator theory. Let $\Omega \cap \mathbf{R}^n$ be an open set; for $u \in L^1(\Omega)$, $\hat{u}$ is the Fourier transform of u . For $\alpha = (\alpha_1, \dots, \alpha_n)$ a multiorder, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ with $D_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$, is the derivative of order α .

The symbols space in $\Omega \times \mathbf{R}^n$ is denoted by $S_{\rho, \delta}^m(\Omega \times \mathbf{R}^n)$. We use the theorem of asymptotic sum in the following form: let $a_j \in S_{\rho, \delta}^{m_j}(\Omega \times \mathbf{R}^n)$, when $\{m_j\}_0^\infty$ is strictly decreasing to $-\infty$ there exists (and is unique determined modulo $S_{\rho, \delta}^{-\infty}(\Omega \times \mathbf{R}^n)$, $a \in S_{\rho, \delta}^m(\Omega \times \mathbf{R}^n)$, such that: for every $k \in \mathbf{N}$, $a - \sum_{j \leq k} a_j \in S_{\rho, \delta}^{m_k}(\Omega \times \mathbf{R}^n)$; the symbol „ a ” is called asymptotic sum of sequence $\{a_j\}_0^\infty$ and we write $a \sim \sum_{j=0}^\infty a_j$.

The spaces, $C_0^\infty(\Omega)$, $C^\infty(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$ are considered with canonical topologies for the locally convex spaces. The notation $L_{\rho, \delta}^m(\Omega) = L_{\rho, \delta}^m(\Omega \times \mathbf{R}^n)$ with $m \in \mathbf{R}$, $\rho, \delta \in [0, 1]$ is used for the space of pseudodifferential operators defined in Ω . Their characteristic property is the pseudolocality, i.e. for every $P \in L_{\rho, \delta}^m(\Omega)$ and for every $u \in \mathcal{E}'(\Omega)$, we have: $\text{sing spt } Pu \subset \text{sing spt } u$.

Definition: The operator $P \in L_{\rho, \delta}^m(\Omega)$ is hypoelliptic if $\text{sing spt } Pu = \text{sing spt } u$, for every $u \in \mathcal{E}'(\Omega)$.

If $P \in L_{\rho, \delta}^m(\Omega)$ and $\rho > \delta$ is properly defined, the expression $e_P(x, \eta) = e^{-i\langle x, \eta \rangle} P_y(e^{i\langle y, \eta \rangle})$ is the symbol of P ; if $P, Q \in L_{\rho, \delta}^m(\Omega)$ are properly supported, then composition $P \circ Q$ is well defined, when $(P \circ Q)(u) = P(Qu)$, $u \in \mathcal{E}'(\Omega)$ and $\sigma_{P \circ Q}(x, \eta) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\eta^\alpha \sigma_P(x, \eta) \cdot D_x^\alpha \sigma_Q(x, \eta)$.

II. The condition of hypoellipticity

Theorem: Let $P \in L_{\rho, \delta}^m(\Omega)$, $\rho > \delta$ and $p(x, \xi)$ is symbol. If A) and B) are satisfied then P is hypoelliptic where:

A) For every compact set $K \subset \Omega$, and for any multiorders α, β there are $C_{K, \alpha, \beta} > 0$ and $R_K > 0$ (constants), such that:

$$|p^{-1}(x, \xi) D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{K, \alpha, \beta} (1 + |\xi|)^{-\rho|\alpha| + \delta|\beta|}$$

for every $x \in K$, and every $\xi \in \mathbf{R}^n$ $|\rho| > R_K$.

B) There is $m' \in \mathbf{R}$ such that: for every compact set $K \subset \Omega$ there is $C_K > 0$ such that:

$$|p^{-1}(x, \xi)| \leq C_K |\xi|^{m'}, \text{ for } x \in K, \{\xi\} \in K, |\xi| \rightarrow +\infty, \xi \in \mathbf{R}^n$$

Proof: It is constructed Q , a parametrix for P , i.e. $Q \circ P - I \in L_{\rho, \delta}^{-\infty}(\Omega)$ since P is hypoelliptic. The symbol q of parametrix satisfies:

$$\sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} q(x, \xi) D_x^{\alpha} p(x, \xi) \sim 1 \text{ (module } S_{\rho, \delta}^{-\infty}(\Omega)).$$

For the short, the left member of this expression is noted by $q \circ p$. Let be $e_0(x, \xi) = p^{-1}(x, \xi)$; for $K \subset \Omega$ a compact set, let be $h \sim 1 - e_0 \circ p$. It is easy to prove that the asymptotic sum exist (for exemple in [1]) and $h \in S_{\rho, \delta}^{-(\rho-\delta)}(\Omega)$. The symbol q of parametrix is $q = \tilde{h} \circ e_0$, when $\tilde{h} \sim 1 + h + h \circ h + h \circ \dots \circ h + \dots$ (modulo $S_{\rho, \delta}^{-\infty}(\Omega)$; we have $\underbrace{h \circ \dots \circ h}_k \in S_{\rho, \delta}^{-k(\rho-\delta)}(\Omega)$)

and hence the asymptotic sum exist and $\tilde{h} \in S_{\rho, \delta}^0(\Omega)$. Let $\tilde{h}_k \sim 1 + h + \dots + \underbrace{h \circ \dots \circ h}_k$; $\tilde{h} \sim \tilde{h}_k + f_k$, $f_k \in S_{\rho, \delta}^{-(k+1)(\rho-\delta)}(\Omega)$ and it is obvious that

$\tilde{h}_k \sim 1 + h \circ \tilde{h}_{k-1} \sim 1 + \tilde{h}_{k-1} \circ h$ (modulo $S_{\rho, \delta}^{-k(\rho-\delta)}(\Omega)$). Changing $\tilde{h}_k \sim \tilde{h} - f_k$, and using the classes of equivalence modulo $S_{\rho, \delta}^{-\infty}(\Omega)$ it results $\tilde{h} \sim 1 + \tilde{h} \circ h \sim 1 + h \circ \tilde{h}$.

In the end, we obtain the expression from the fact that q is symbol for Q : $q \circ p \sim (\tilde{h} \circ e_0) \circ p \sim \tilde{h} \circ (e_0 \circ p) \sim \tilde{h} \circ (1 \sim h) \sim 1$ (modulo $S_{\rho, \delta}^{-\infty}(\Omega)$).

III. The operators with constant coefficients.

Let $P(D) = \sum_{|\alpha| \leq m} c_{\alpha} D^{\alpha}$ be a differential operator with constant coefficients; it is obvious that $P \in L_{1,0}^m(\Omega)$, when $P(D) : C_0^{\infty}(\mathbf{R}^n) \rightarrow C_0^{\infty}(\mathbf{R}^n)$.

Corollary: Let $\rho > 0$ such that is satisfied the condition

$$a1) : \left| \frac{D_{\xi}^{\alpha} P(\xi)}{P(\xi)} \right| \leq C |\xi|^{-\rho|\alpha|}, \text{ when } C > 0 \text{ is a constant for every}$$

$\xi \in \mathbf{R}^n$ and $|\xi| \rightarrow +\infty$, $P(\xi) \neq 0$. Then $P(D)$ is hypoelliptic.

Remark: In [2] and [3] is proved the converse assertion. In fact, the hypoelliptic operators with constant coefficients are characterized by a1).

IV. Formal hypoelliptic operators.

Definition: Let R and S be two differential operators with constant coefficients: R is weaker than S (S is stronger than R) and is noted $R < S$,

if there exists $C > 0$ such that $\tilde{R}(\xi)/\tilde{S}(\xi) \leq C$ for every $\xi \in \mathbf{R}^n$, where $\tilde{R}(\xi) = (\sum_{\alpha} |K^{\alpha}(\xi)|^2)^{1/2}$. If $R < S < R$, we say that R and S are equivalent.

Let $P(x, D) = \sum_{|\alpha| \leq m} a_{\alpha}(x) D^{\alpha}$, $a_{\alpha} \in C_0^{\infty}(\Omega)$ be a differential operator with variable coefficients; $P(x, D)$ is said to be formal hypoelliptic if:
 i) for every $x \in \Omega$, the operators with constant coefficients $P(x, D)$ are hypoelliptic, and
 ii) the operators with constant coefficients $P(x, D)$ and $P(y, D)$ are equivalent, for every $x, y \in \Omega$.

Remarks: 1. The differential operators R weaker than (or stronger than) P form a linear space.

2. It results that the equivalent operators form a linear space.

3. It is possible, to assume in definition that condition i) is satisfied only for $x_0 \in \Omega$ considered fixed from the fact that hypoellipticity is invariant under equivalence of operators with constant coefficients.

Theorem: Let $P(x, D)$ be a formal hypoelliptic operator with $C^{\infty}(\Omega)$ coefficients; then $P(x, D)$ is hypoelliptic if A) and B) are satisfied

Proof: Let $x, y \in \Omega$. The polynomials $P(x, \xi)$ and $P(y, \xi)$ can have common kernel. Let $x_0 \in \Omega$ be fixed. Then $P(x_0, \xi)$ and $P(y, \xi)$ are equivalent in ξ (in other case, we have $\tilde{P}(y, \xi_0)/\tilde{P}(y, \xi) \rightarrow +\infty$ because $\xi_0 \in \mathbf{R}^n$, $P(x_0, \xi_0) = 0$).

If the coefficients of $P(x_0, \xi)$ in x_0 are simultaneously zero, then there is a contradiction to the equivalence of $P(x_0, \xi)$ and $P(y, \xi)$.

Hence the polynomials $P(x, \xi)$ and $P(y, \xi)$ are simultaneously zero in their arguments.

Let $\mathcal{S}$ be the space of equivalent polynomials with $P(x_0, D)$, where $x_0 \in \Omega$ is fixed and let $\{P_j\}_1^N$ be a basis in $\mathcal{S}$ note containing $P(x_0, D)$.

We may write $P(x, D) = \sum_{j=1}^N c_j(x) P_j(D)$, $c_j \in C^{\infty}(\Omega)$. If for $\tilde{x} \in \Omega$ we have

$P(\tilde{x}, \xi) = 0$, then $c_j(\tilde{x}) = 0$ for $j = 1, \dots, N$; This is a contradiction because $P_j(\xi) = 0$, $j = 1, \dots, N$. Thus the zeros of $P(x, \xi)$ are the zeros of $\{P_j\}_1^N$. For every $x \in \Omega$ exist $p^{-1}(x, \xi)$ for $\xi \in \mathbf{R}^n$ if $P_j(\xi) \neq 0$, $j = 1, \dots, N$.

Proof of condition A): Let $K \subset (\Omega)$ be a compact set and $x \in K$, $\xi \in \mathbf{R}^n$ such that $P_j(\xi) \neq 0$, $j = 1, \dots, N$;

$$E = |p^{-1}(x, \xi) D_x^{\beta} D_{\xi}^{\alpha} p(x, \xi)| = |p^{-1}(x, \xi) \sum_{j=1}^N D_x^{\beta} c_j(x) D_{\xi}^{\alpha} P_j(\xi)| \leq$$

$$\leq \sum_{j=1}^N |D_x^{\beta} c_j(x) D_{\xi}^{\alpha} P_j(\xi)| \left| \sum_{j=1}^N c_k(x) P_k(\xi) \right|.$$

Let $C_{K, \beta} > \sup_{j=1, \dots, N} |D_x^{\beta} c_j(x)|$ and $\delta \in (0, 1)$ such that for $|\xi| \rightarrow +\infty$

$$C_{K, \beta} \leq C_{K, \beta} |\xi|^{\delta |\beta|}$$

But for every $r = 1, \dots, N$ there is $C_{K,r} > 0$ such that $\left| \sum_{k=1}^N c_k(x) \cdot P_k(\xi) \right| > \geq C_{K,r} |P_r(\xi)|$ for every $x \in K$, $\xi \in \mathbf{R}^n$. This inequality is true even if the coefficients of $P_r(\xi)$ are zero in x_0 because the polynomials have common degrees with $P(x, \xi)$. Hence

$$\begin{aligned} E &\leq C_{K,\beta} |\xi|^{\delta|\beta|} \sum_{j=1}^N |D_{\xi}^{\alpha} P_j(\xi)| \left| \sum_{k=1}^N c_k(x) P_k(\xi) \right| \leq \\ &\leq C_{K,\beta} |\xi|^{\delta|\beta|} \cdot \sum_{j=1}^N |D_{\xi}^{\alpha} P_j(\xi)| / |P_j(\xi)| \leq C_{K,\alpha,\beta} |\xi|^{-\rho|\alpha| + \delta|\beta|} \end{aligned}$$

here ρ is smaller than the constants ρ_j of operators $P_j(D)$. Proof of condition B): We use Tarski-Seidenberg-Puiseux theorem; let $K \subset \Omega$ be a compact set and $\xi \in \mathbf{R}^n$ such that $P_j(\xi) \neq 0$, $j=1, \dots, N$, $M_R = \{(x, \xi) \in K \times \mathbf{R}^n; -\prod_{j=1}^N |P_j(\xi)|^2 \leq -\frac{1}{R}; |\xi| = R\}$ for R sufficiently large. it is clear that M_R is nonempty and let $\mu(R) = \inf_{(x,\xi) \in M_R} |P(x, \xi)|$. We may use the supremum in definition of $\mu(R)$ and we have the asymptotic development of $\mu(R)$: $\mu(R) = AR^a (1 + o(1))$ for $R \rightarrow \infty$, where $A > 0$ and a is a rational constant. Particularly, $|P(x, \xi)| \geq C_K |\xi|^a$ for $|\xi| \rightarrow +\infty$ with $C_K > 0$ and hence $|P^{-1}(x, \xi)| \leq C_K |\xi|^{-m'}$ for $|\xi| > R_K$.

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Grupul Școlar nr. 10
București



Profesor GHEORGHE VRĂNCEANU

UN PRECURSOR AL GEOMETRIEI DIFERENȚIALE GLOBALE

Înzestrat cu o intuiție de excepție, regretatul nostru profesor și îndrumător Gheorghe Vrănceanu a elaborat, printr-o intensă și îndelungată muncă de cercetare științifică, o serie de concepte cu caracter global, care rămân și astăzi actuale prin profunzimea și ineditul lor. Aspectul global în opera lui Vrănceanu este uneori greu de surprins, datorită formalismului clasic, de tip Levi-Civita, pe care l-a folosit cu predilecție. De aceea valorificarea moștenirii științifice lăsată de Vrănceanu este pe cât de necesară, pe atât de dificilă.

Elevii săi au vii în memorie multe dintre ideile pe care le împărtășea cu generozitate la cursurile și seminariile sale științifice. Pe de altă parte trebuie subliniat interesul crescând de care se bucură opera sa în cadrul unor manifestări științifice prestigioase din țară și de peste hotare. Astfel, la Colocviile de Geometrie Diferențială de la Würzburg (iunie 1979) și Budapesta (septembrie 1979), „Lecțiile de Geometrie Diferențială” ale lui G. Vrănceanu au fost citate de mulți participanți în legătură cu spațiile neolome, cu conexiunile proiective și cu alte probleme. În alocuțiunea de deschidere a Colocviului de la Budapesta, profesorul G. Soos ar evocat în cuvinte emoționante personalitatea lui G. Vrănceanu, întreaga asistență păstrându-i un minut de reculegere. Apariția recentă a volumului „Lucrările Conferinței Naționale de Spații Neolome” la Editura Acad. R.S.R. (1979) este o mărturie pentru înalta prețuire de care se

bucură demiurgul spațiilor neolome, cum îl numea pe Vrănceanu, matematicianul și poetul Dan Barbilian.

Aici, la București—la Universitatea pe care a slujit-o cu devotament timp de patru decenii — se continuă astăzi cercetările științifice inspirate din lucrările lui Vrănceanu la un nalt nivel științific.

Scopul lucrării de față este de a ilustra prin cîteva exemple cum au apărut în opera lui Vrănceanu anumite idei și concepte de geometrie diferențială globală. Pentru o analiză amănunțită a operei lui G. Vrănceanu pînă la 1960 se poate consulta lucrarea lui C. Teleman [8,] iar pentru teoria spațiilor neolome se poate consulta lucrarea [5].

Pentru a ne face cît mai bine înțelegi, vom folosi un formalism modern

Fie M o varietate diferențială de clasa C^∞ de dimensiune n , conexă și paracompactă. Toate obiectele geometrice considerate pe M în cele ce urmează sînt de clasă C^∞ .

Notăm $\mathcal{F}(M)$ inelul funcțiilor diferențiabile pe M , $\mathcal{X}(M)$ algebra Lie a cîmpurilor de vectori pe M , $T(M)$ fibrarea principală a reperelor liniare pe M , $T^*(M)$ fibrarea tangentă a lui M și $T^*(M)$ fibrarea sa cotangentă. Grupul structural al acestor fibrări este grupul general liniar real $GL(n)$, iar $T(M)$ și $T^*(M)$ sînt fibrări vectoriale asociate în mod canonic fibrării principale $L(M)$. Este bine cunoscut faptul că teoria G — structurilor, o realizare remarcabilă a anilor 50, a reușit să dea o tratare unitară a tuturor structurilor geometrice clasice și a condus chiar la structuri geometrice, noi. Dacă G este un subgrup închis al lui $GL(n)$, atunci o G -structură pe varietatea diferențiabilă M este prin definiție o reducere a fibrării $L(M)$ la grupul structural G .

1. Elemente de G -structuri în opera lui G. Vrănceanu

În contextul de mai sus, pentru noi este evident faptul că clasa metricilor riemanniene pe varietatea diferențiabilă M este în bijectie cu clasa $O(n)$ — structurilor pe M , care sînt reduceri ale lui $L(M)$ la grupul ortogonal real $O(n)$. Ori această proprietate a fost sesizată de Vrănceanu încă din 1926, cu mult înainte de constituirea teoriei G -structurilor. În adevăr, Vrănceanu a introdus de la bun început și a folosit cu consecvență noțiunea de *transformare de congruențe*, după cum el însuși explică în nota istorică din [5] și după cum rezultă cu claritate din lucrările [9], [10] [11]. Un sistem de congruențe ds ($h^h = 1, \dots, n$), este un cîmp local de corepere în fibrarea cotangentă $T^*(M)$.

Vrănceanu observă că orice metrică riemanniană g pe M este unic determinată de existența unor sisteme de congruențe care se transformă între ele prin ortogonalitate :

$$(1) \quad d\bar{s}^h = c_k^h ds^k, \quad c_k^h c_l^h = \delta_{kl} \quad (h, k, l = 1, \dots, n)$$

Prin trecere la cîmpurile derepere în $T(M)$ duale sistemelor (ds^h) de congruențe ortogonale, constatăm, că (c_k^h) din (1) constituie funcțiile de tranziție ale acelei reduceri a fibrării $L(M)$ la grupul structural $O(n)$ care corespunde metricii g .

Calculul diferențial absolut pe congruențe, ale cărui scop este de a vedea cum se comportă anumite obiecte geometrice în raport cu transformările de congruențe, joacă un rol central în opera lui Vrănceanu, de la primele sale lucrări până la „Lecțiile de Geometrie Diferențială” [14]. Acest nou tip de calcul i-a permis să studieze în detaliu noțiunea de spațiu neolonom pe care a descoperit-o în 1926 (a se vedea [9] [10])

În limbaj actual, un spațiu neolonom de dimensiune m pe varietatea diferențiabilă M de dimensiune $n > m$, este o distribuție neintegrabilă D de rang m pe varietatea M . Dacă notăm cu H grupul transformărilor din $GL(n)$ de forma

$$(2) \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad a \in GL(m), \quad c \in GL(n-m),$$

atunci spațiul neolonom D apare ca o H -structură pe varietatea diferențiabilă M . Cum grupul $H' = GL(m) \times GL(n-m)$ este o retracție de deformare pentru grupul H al transformărilor (2), rezultă existența unei H' -structuri pe varietatea diferențiabilă M , deci a unei distribuții D' de rang $n-m$, complementară distribuției D . Evident, reducerea fibrării $L(M)$ la grupul structural H' nu este unică, cu alte cuvinte distribuția complementară D' nu este unic determinată de spațiul neolonom D .

Menționăm că Vrănceanu a insistat de nenumărate ori în legătură cu caracterul grupal al noțiunilor de spațiu Riemann și de spațiu neolonom. Explicațiile care le-a dat în această privință prefigurează noțiunea de G -structură. Mai mult, condiția de neintegrabilitate pe care a impus-o spațiilor neolome coincide cu neintegrabilitatea în sensul G -structurilor. Pentru G -structurile cu grupul structural complet reductibil G , Vrănceanu a demonstrat în 1941 următoarea teoremă [12]: *Dacă un grup Pfaff complementare*

$$(3) \quad ds^h = 0, \quad ds^\alpha = 0, \quad h = 1, \dots, m; \quad \alpha = m+1, \dots, n$$

atunci există un transport paralel (o conexiune afină) dacă sistemele nu au combinații integrabile.

Această teoremă a fost utilizată de mai mulți matematicieni în cercetări recente asupra distribuțiilor. În aceeași lucrare [12] G. Vrănceanu descoperă un obiect geometric în teoria spațiilor neolome, care nu are caracter tensorial, dar este intim legat de tensorul de integrabilitate asociat spațiului neolonom. Acest obiect geometric, cunoscut în literatură sub numele de *obiectul geometric al lui Vrănceanu*, a fost caracterizat global de R. Caddeo [1] prin formula următoare

$$(4) \quad A(X, Y; Z) = V[V[V'X, V'Y], VZ] - V[V'[X, VZ], V'Y] \\ - V[V'X, V'[Y, VZ]]$$

unde V și V' sînt proiectorii asociați distribuțiilor complementare D și D' . Pe de altă parte, în 1957 A.G. Walker a asociat unei structuri aproape complexe o derivată torsională. Pornind de la obiectul geometric al lui Vranceanu, Th. Hangan [2] a generalizat derivata torsională a lui Walker pentru G -structuri cu grup complet reductibil. Această generalizare i-a permis lui Th. Hangan să arate legătura dintre obiectul geometric al lui Vranceanu și derivata torsională a lui A.G. Walker.

2. Caracterul inedit al conexiunii Vranceanu

Fiind date două distribuții complementare D și D' de rang m , respectiv $n-m$ pe varietatea diferențiabilă M , fibrarea tangentă $T(M)$ admite în mod canonic grupul structural H' și se identifică cu suma Whitney $D \oplus D'$. Clasa perechilor (D, D') de distribuții complementare pe M este în bijecție cu clasa structurilor aproape produs ale lui M' definite prin $J = V + V'$.

Vom comenta acum construcția globală dată de S. Ianuș în lucrarea [3], pentru conexiunile lui Schouten și Vranceanu asociate unei perechi de distribuții complementare (D, D') și unei conexiuni oarecare ∇^0 pe varietatea M .

Folosind proiectorii V și V' definim mai întîi următoarele conexiuni $\bar{\nabla}$ și $\bar{\nabla}'$ în sensul lui Koszul pe fibrările vectoriale D , respectiv D' :

$$(5) \quad \bar{\nabla}_X Y = V \nabla_X^0 Y, \quad \bar{\nabla}'_X Y' = V' \nabla_X^0 Y', \quad X \in \mathcal{X}(M), \quad Y \in \mathcal{S}(D), \quad Y' \in \mathcal{S}(D')$$

unde $\mathcal{S}(D)$ și $\mathcal{S}(D')$ sînt spațiile de secțiuni ale fibrărilor D și D' . Se obține următoarea conexiune liniară ∇^1 a lui M :

$$(6) \quad \nabla_X^1 Y = \bar{\nabla}_X V Y + \bar{\nabla}'_X V' Y, \quad X, Y \in \mathcal{X}(M),$$

numită *conexiunea lui Schouten*. Reciproc, fiind date două conexiuni oarecare $\bar{\nabla}$ și $\bar{\nabla}'$ prima pe D și a doua pe D' , formula (6) definește o conexiune ∇^1 pe $T(M) = D \oplus D'$, deci o conexiune liniară a varietății diferențiabile M .

Introducem acum două multiplicări pe spațiul vectorial $\mathcal{X}(M)$, prin:

$$(7) \quad [X, Y]_V = [VX, VY]; \quad [X, Y]_{V'} = [V'X, V'Y] \quad X, Y \in \mathcal{X}(M).$$

Atunci fiecărui cîmp de vectori $X \in \mathcal{X}(M)$ i se asociază cîte o derivare pe $\mathcal{F}(M)$ punînd

$$(8) \quad X(f)_V = VX(f), \quad X(f)_{V'} = V'X(f), \quad f \in \mathcal{F}(M)$$

Definind aplicațiile :

$$\nabla : \mathcal{X}(M) \times \mathcal{S}(D) \rightarrow \mathcal{S}(D), (X, Y) \rightarrow \nabla_X Y = V \nabla_{VX}^0 Y, \quad (9)$$

$$\nabla' : \mathcal{X}(M) \times \mathcal{S}(D') \rightarrow \mathcal{S}(D'), (X, Y') \rightarrow \nabla'_X Y' = V' \nabla_{V'X}^0 Y',$$

constatăm că ambele sînt $\mathcal{F}(M)$ — liniare în raport cu argumentul X . În ceea ce privește al doilea argument ambele aplicații ∇ și ∇' sînt aditive și avem în plus

$$\nabla_X(fY) = f \nabla_X Y + X(f)_V Y, \quad X \in \mathcal{X}(M), \quad Y \in \mathcal{S}(D), \quad f \in \mathcal{F}(M), \quad (10)$$

$$\nabla_{VX} Y = \nabla_X Y$$

precum și formule analoage referitoare la ∇' .

Construim o nouă conexiune liniară ∇^2 pe $T(M)$, anume :

$$\nabla_X^2 Y = \nabla_X VY + \nabla'_X V'Y + V'[VX, V'Y] + V[V'X, VY], \quad X, Y \in \mathcal{X}(M) \quad (11)$$

numită *conexiunea lui Vrănceanu*. O aplicație ∇ de forma (9), care este $\mathcal{F}(M)$ — liniară în argumentul X , aditivă în Y și satisface condițiile (10), se numește V -conexiune pe distribuția D ; în mod analog se introduce noțiunea de V' -conexiune pe distribuția complementară D' . Atunci, pentru o V -conexiune oarecare ∇ și o V' — conexiune ∇' , putem construi totdeauna o conexiune liniară ∇^2 pe varietatea diferențiabilă M cu ajutorul lui (11). Prin urmare, conexiunea lui Vrănceanu conduce la o noțiune nouă — aceea de V — conexiune — pe cînd conexiunea lui Schouten nu depășește cadrul definiției lui Koszul a conexiunilor pe fibrări vectoriale. Mai mult, oricărei V -conexiuni ∇ îi putem asocia operatorul de curbură Γ prin :

$$(12) \quad \Gamma(X, Y, Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]} Z - V[V'[X, Y] V, Z].$$

Remarcăm că Γ este $\mathcal{S}(D)$ — valuat și $\mathcal{F}(M)$ — liniar în raport cu argumentele sale $X, Y \in \mathcal{X}(M), Z \in \mathcal{S}(D)$.

Geometria riemanniană a spațiilor neolonome construită de G. Vrănceanu conține de asemenea fapte remarcabile, dar considerăm că aspectele prezentate aici sînt suficiente pentru a arăta generalitatea conceptelor ce zvorăsc din cercetarea spațiilor neolonome.

4. Aspectul global al noțiunii de conexiune constantă

Deși secundară în raport cu problematica spațiilor neolonome, conceptul lui Vrănceanu de spații cu conexiune constantă a condus totuși la cercetări fructuoase în țară și peste hotare.

După G. Vrănceanu o conexiune liniară ∇ a spațiului real n -dimensional R^n este constantă [13] [14] dacă componentele sale Γ_{jk}^i în raport cu harta identitate a lui R^n sînt constante. O conexiune liniară ∇ a lui R^n este constantă dacă și numai dacă ∇ este o conexiune invariantă în raport cu grupul translațiilor lui R^n . Vrănceanu numește „preferențiale” transformările afine de coordonate ale lui R^n , deoarece acestea mențin proprietatea componentelor Γ_{jk}^i de a fi constante.

Pe de altă parte Γ_{jk}^i pot fi interpretate drept constantele de structură ale unei algebre reale n -dimensionale A într-o bază a sa $b = (b_1, \dots, b_n)$ aume:

$$(13) \quad b_j b_k = \Gamma_{jk}^i b_i$$

Există deci o bijecție $\nabla \rightarrow A$ între clasa spațiilor cu conexiune constantă și clasa algebrelor reale de dimensiune finită. Se naște astfel problema de a transpune în limbaj algebric proprietățile conexiunii constante ∇ și de a da semnificație geometrică proprietăților algebrei A . De exemplu, conexiunea constantă ∇ este simetrică și plată dacă și numai dacă algebra A este comutativă și asociativă. Aceste considerații sînt susceptibile la o globalizare naturală, după cum a arătat I. Popovici în [6]. Fie (M, ∇) un spațiu cu conexiune liniară, de clasă C^∞ și de dimensiune n . Presupunem că fiecare punct $p \in M$ are o vecinătate deschisă U_p pe care acționează un grup local G_p de transformări, abelian și simplu tranzitiv, astfel încît ∇ să fie conexiune invariantă în raport cu G_p . Atunci spațiul vectorial n -dimensional al cîmpurilor de vectori pe U_p , care sînt invariante în raport cu grupul G_p , admite o structură canonică de algebră reală $A(\nabla, G_p)$ cu multiplicarea $(X, Y) \rightarrow \nabla_X Y$. Dacă în plus toate algebrele $A(\nabla, G_p)$, $p \in M$, sînt izomorfe cu o algebră A , se spune că tripletul (M, ∇, A) este un *spațiu cu conexiune constantă*.

Remarcăm că fiecare grup G_p poate fi realizat ca grup local de translații în raport cu o hartă convenabilă $\mathcal{H}$, caz în care componentele Γ_{jk}^i ale conexiunii ∇ să fie constante și să definească constantele de structură ale algebrei A într-o bază $b^\mathcal{H}$. Mai mult, se poate construi un atlas α al lui M format din hărți H verificînd proprietatea de mai sus; se poate presupune în plus că bazele $b^\mathcal{H}$, deci componentele constante Γ_{jk}^i , nu depind de $\mathcal{H}$. Atlasul L induce o structură analitică reală pe spațiul cu conexiune liniară (M, ∇) , presupus inițial de clasa C^∞ .

Rezultă că definiția spațiului cu conexiune constantă (M, ∇, A) înglobează toate aspectele de natură locală menționate de Vrănceanu, cu excepția faptului că atlasul α poate fi afin sau nu. În acest mod apar și transformări preferențiale de coordonate care nu sînt afine și a căror existență este dovedită pe bază de exemple.

4. Metrica lui Vrănceanu și subvarietăți în spații euclidiene

Teoria scufundării unei varietăți Riemann într-un spațiu euclidian este una din teoriile cele mai dezvoltate în geometria diferențială.

Profesorul G. Vrănceanu și-a înscris numele în această teorie din 1930 cînd a dat o formulă celebră cunoscută sub numele de metrica lui Vrănceanu. Vom prezenta această formulă în terminologia lui I. Popovici,

R. Iordănescu și A. Turtoi [7]. Fie M o varietate Riemann n -dimensională care admite o scufundare într-un spațiu euclidian E^m , $f : M \rightarrow E^m$. Pentru orice $x \in M$ să notăm cu T'_x spațiul ortogonal la $f_*(T_x M)$, un reper normal N în x este o bază ortonormată $(N_{n+1}, \dots, N_m)$ a lui T'_x . Se arată că mulțimea $P = \{(x, N) | x \in M, N \text{ reper în } x\}$ este o varietate diferențiabilă pe care acționează liber grupul ortogonal $O(m-n, R)$ și astfel se obține fibrarea principală normală $(P, M, O(m-n, R))$ asociată scufundării $f : M \rightarrow E^m$.

Să notăm $\partial = (\partial_{n+1}, \dots, \partial_m)$ un câmp de repere normale definit pe o vecinătate de coordonate U a varietății M .

Atunci rezultă 1-formele $\{\psi_{pq}\}$ și câmpurile de tensori $\{\varphi_p\}$ de tip $(0,2)$ date prin formulele :

$$(14) \quad \psi_{pq}(X) = -\psi_{qp}(X) = \langle \partial_p \cdot X, \partial_q(x) \rangle$$

(15)

$$\varphi_p(X, Y) = -\langle f_*(X), \partial_p \cdot Y \rangle$$

unde $x \in U$ și $X, Y \in T_x M$

Câmpurile de tensori $\{\varphi_{n+1}, \dots, \varphi_m\}$ definesc a doua formă fundamentală a lui M iar $\{\psi_{pq}\}$, $n+1 \leq p, q \leq m$, au fost numite de Vrănceanu torsiuni prin analogie cu noțiunea de torsiune din teoria curbelor în E^3 .

Dacă se introduce pe U coordonatele $(x^1, \dots, x^n)$ atunci scufundarea f se exprimă prin formulele

$$(16) \quad y^a = f^a(x^1, \dots, x^n) \quad a = 1, \dots, m$$

G. Vrănceanu a asociat formulelor (16) funcțiile

$$(17) \quad y^a = f^a(x^1, \dots, x^n) + \sum_{p=n+1}^m \partial_p^a x^p$$

care ne dau un sistem de coordonate curbilini $(x^1, \dots, x^m)$ într-un deschis V din E^m ce conține deschisul U din M dacă se face identificarea lui M cu $f(M)$. Ideia lui Vrănceanu de a introduce coordonatele $(x^{n+1}, \dots, x^m)$ prin formula (17) l-a condus în mod natural la metrica care-i poartă numele (utile în multe cazuri de scufundări). Înlocuind (y^a) date de formulele

(17) în metrica euclidiană $ds^2 = \sum_{a=1}^m (dy^a)^2$. G. Vrănceanu a obținut metrica

care-i poartă numele,

$$(18) \quad ds^2 = d\sigma^2 - 2x^p \varphi_p + x^p x^q \varphi_{pq} + 2x^p dx^q \psi_{pq} + \sum_{p=n+1}^m (dx^p)^2$$

unde $d\sigma^2$ este metrica lui M , ψ_{pq} și φ_p sînt date de (14) respectiv (15)

iar φ_{pq} sînt cîmpurile de tensori simetrici definite prin

$$(19) \quad \varphi_{pq}(X, Y) = \langle \partial_p X, \partial_q Y \rangle \quad X, Y \in T_x M$$

Unul din avantajele metricii lui Vrănceanu este că, anulînd tensorul de curbura asociată metricii se obțin în mod unitar ecuațiile lui Gauss, Codazzi și Ricci-Kühne.

Metoda lui Vrănceanu a fost valorificată de L. Einsenhart în bine cunoscutul tratat de geometrie „Riemannian Geometry”, Princeton, 1949 și de mulți alți geometri.

G. Vrănceanu a asociat formelor de torsiune $\{\psi_{pq}\}$ o conexiune $\tilde{\nabla}$ numită conexiunea torsiunilor, (care este același lucru cu conexiunea în fiabratul normal definită în [4], formulată în limbaj de fibrări de I. Popovici, R. Iordănescu, A. Turtoi în lucrarea [7]. O scufundare $f: M \rightarrow E^n$ este numită fără torsiuni de G. Vrănceanu dacă conexiunea $\tilde{\nabla}$ are curbura zero. G. Vrănceanu demonstrează ([14] vol. IV) că scufundarea este fără torsiuni dacă și numai dacă în jurul fiecărui punct există un sistem de congruențe în care tensorii φ_p au forma diagonală. Acest rezultat a fost regăsit prin metode recente și utilizate de unii matematicieni străini specialiști în teoria subvarietăților (B. Chen, M. Okumura, K. Ogiue, K. Yano etc.)

În ultimii ani G. Vrănceanu a manifestat un interes deosebit pentru unele noțiuni și rezultate fundamentale din geometria globală pentru care a dat demonstrații noi de mare simplitate sau le-a ilustrat cu aplicații interesante dintre care vom selecta cîteva. Astfel, într-o lucrare recentă [16] G. Vrănceanu a asociat unei subvarietăți închise V^n în E^m o subvarietate V^{m-1} definită local prin ecuațiile

$$(20) \quad Z^a = \partial_p^a x^p \quad a = 1, \dots, m; \quad p = n+1, \dots, m$$

$$(21) \quad (x^{n+1})^2 + \dots + (x^m)^2 = 1$$

și a arătat că invariantul lui Kuiper (vezi [14] vol. IV), în cazul că V^n are torsiunile nule, este raportul dintre volumul lui V^{m-1} și volumul sferei S^{m-1} definită de (21).

Fie acum o curbă C orientată și închisă situată pe o suprafață închisă S . G. Vrănceanu a arătat (1971) că există un întreg l astfel încît ([14] vol IV)

$$(22) \quad \int_C K_g ds + \iint_D K d\sigma = 2\pi l$$

unde K_g este curbura geodezică a curbei C , K este curbura Gauss a suprafeței S iar D este domeniul pe S mărginit de C și lăsat la stînga cînd se parcurge curba în sensul orientării fixate.

Să presupunem că curba C este situată pe sfera standard S^2 din E^3 . Fie $\Delta\theta$ variația unghiului θ a unui vector V cu un vector fixat când V se deplasează în lungul lui C . O teoremă a lui Levi-Civita afirmă că dacă $\Delta\theta = 0$ atunci C divide S^2 în două regiuni de arii egale. Vrănceanu a demonstrat că teorema lui Levi-Civita este un corolar al formulei (22) și a arătat cum se poate extinde la suprafețe de gen oarecare ([14] vol. IV). Într-o lucrare din 1975 ([14] vol. IV) folosind noțiunea de exces al unei curbe (introdusă de Willmore) G. Vrănceanu arată că în formula (22) avem $l = 1$ dacă și numai dacă D este un domeniu simplu conex. Mai mult, arată că dacă parcurgând C pe sensul fixat avem n domenii simplu conexe la stînga și m domenii simplu conexe la dreapta, atunci $l = n - m$. G. Vrănceanu a arătat că întregul l depinde de numărul de puncte duble ale curbei derivate C' asociate curbei C . Pe de altă parte, numărul de puncte duble ale curbei C' este egal cu numărul de cupluri (P, Q) de puncte de pe C în care tangentele sînt paralele și au același sens, după cum a arătat B. Segre în 1968.

În sfîrșit, menționăm demonstrația simplă și aplicațiile interesante făcute de G. Vrănceanu asupra invariantului Călugăreanu ([14] vol. IV).

Vedem astfel că opera lui Vrănceanu este organic integrată în cercetările geometriei diferențiale moderne.

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S. IANUȘ, I. POPOVICI



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